Mathematical Aspects of Classical Mechanics

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From a mathematical point of view classical mechanics combines a **great variety of mathematical objects**, such as differential equations, manifolds, Lie groups and Lie algebras, variational calculus, symplectic geometry and ergodic theory. In physics motion of objects is described by differential equations, but as we will see the solutions of these equations, i.e. the trajectories of our objects in phase space, are integral curves of a vector field defined on phase space. This vector field is determined by a differential two form (called *symplectic form*) and the energy function (called *Hamiltonian function*). We will therefore find that phase space actually is a symplectic manifold.

Organization of the seminar

When: Mondays 2:15 pm

Where: Most likely on Zoom.

Language: English

Evaluation: Participants will give a graded 90-minutes talk, write a summary of the talk that will be uploaded to the homepage of the seminar and actively participate during the seminar talks.

Meet us (most likely virtually): One or two weeks before your talk to discuss your plan of the talk and to clarify questions.
Please contact the respective organiser of the talk via mail.

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List of Topics

1 General Background

Topic 1 : Newtons law and phase space (talk with Johanna)

Newtons law is the foundation of classical mechanics. It is a second order ordinary differential equation and therefore its solutions are determined by fixing initial position and momentum. These two quantities span the phase space and time evolution of a physical system given by a curve in phase space. In this talk we introduce the physical concepts of phase space, phase flow and prove Liouville's theorem as well as have a look at symplectic linear algebra which is the mathematical structure behind these concepts. We first introduce the notions of motion, velocity, acceleration and Newtons law according to [Arnold, p. 7-8]. From there on we restrict to conservative sytems i.e. $F(x) = -\nabla V(x)$ for some potential V. Following [dS, chapter 18.2] rewrite Newtons law as a system of first order differential equations and express it in terms of the total energy $H = \frac{1}{2}|v|^2 + V$. Observe that this system is a special case of the Hamilton equations. Continue by introducing phase flow and phase space ([Arnold, page 68-70]) and prove Liouville's theorem ([Bm, page 1137]). The matrix J provides phase space with a symplectic structure. In the last part we shall introduce the general defintion of symplectic vector spaces following [dS, chapter 1.1 and 1.2].

Topic 2 : Calculus of variations and Lagrangian formalism (talk with Valerio)

This part of the course is devoted to the introduction of Lagrangian and Hamiltonian formalism for conservative systems. It turns out that many physical laws can be expressed as "variational principles", that is in terms of critical points for certain functionals. In particular we focus on the Lagrangian Action Functional through which we derive the Euler-Lagrange equation. Changing for a moment the scenario we start to deal with the Hamiltonian formalism. Legendre duality will show that in fact these two approaches are two sides of the same medal. The main reference for this talk is Arnold's book [Arnold, §3].

Topic 3 : Basics on manifolds (talk with Anna-Maria)

The purpose of this talk is to introduce the concepts of a differentiable manifold and its tangent bundle. We will in particular learn that the configuration space of a system with constraints is a differentiable manifold.
We will mainly follow Chapter 4, that is called *Lagrangian mechanics on manifolds*, in [Arnold, p. 75 - 88] for this talk. We start by recalling the definition of *a system with constraints*. Then move on to the definition of a *differentiable manifold* including some examples. With the definition of a differentiable manifold we are now able to

look at the *tangent space* and the *tangent bundle*. This knowledge quickly extends to the definition of a *Riemannian manifold* and *the derivative map*. Finally, we apply all this knowledge and present a *Lagrangian system* in the more general framework of manifolds. The talk will end by considering a *natural Lagrangian* that is a Lagrangian function that is equal to the difference between kinetic and potential energies.
More details on these topic may be found in [Lee, Chapter 1-3] or [Tu, Chapter 2-3].

Topic 4 : Rigid body (talk with Valerio)

The rigid body motion is one of the most interesting and studied example of Lagrangian System as well as one of the most applied mathematical scheme (also outside the boundaries of Classical Mechanics). In this part of the program we look in detail at the standard configurations and some example. This is mainly in [Arnold, §6].

Topic 5 : Symmetries and Noether's theorem (talk with Valerio)

It is well known from a physical point of view that symmetries of the system simplify the equations of the same. The main goal is to translate this correspondence between symmetries and conserved quantities in a mathematical language (Noether's theorem).

Topic 6 : Arnold's theorem (talk with Valerio)

We take the first steps in the huge world of integrable systems. It will be ambitious but not too, to prove the Arnold-Liouville theorem in the one dimensional case.

2 Applications

Topic : Magnetic Systems (talk with Anna-Maria)

As another example of applications of Newton's law, we consider a charged particle moving in a magnetic field. Then the Lorentz force is acting on the particle. We will take a look at the Lagrangian and Hamiltonian formalism of the Lorentz force. The first example could be the constant magnetic field.
Some references are the book *Advanced Mechanics* by S. G. Rajeev [Ra], the lecture notes [GW], the book by J.-L. Basdevant [Ba].
The first talk on this topic starts with describing the Lorentz force in $\mathbb{R}^3$ in terms of the Lagrangian and Hamiltonian formalism including some examples of the magnetic field. One question we may ask ourselves is whether there exists a magnetic field so that the motion of the particle is confined in some region. For this question we consider first the integrable case and use Noether's theorem to note that we need a symmetry. Second we consider the confinement in the non-integrable case. In the non-integrable case, we

may prove [Ma, Thm. 1].
The second talk on this topic deals with the twisted symplectic form. This describes the magnetic systems on manifolds not just $\mathbb{R}^3$. We define the magnetic form on a symplectic manifold and see how the Lorentz force can be described in these terms. We will prove an equivalence for curves that are solutions of Newton's equation.

Topic : Chaos with the double pendulum (talk with Anna-Maria)

The main sources for this talk are *An Introduction to Dynamical Systems and Chaos* [La, §12] and *Chaos* [KJH, §5].
In this talk we start by introducing chaos theory in the context of dynamical systems [La, intro §12]. In common language, chaos means *a state of disorder*. There is also a more precise definition of chaos going back to *Devaney* [La, Def.12.9]. As a first example of a chaotic map, look at [La, Example 12.1]. In [La, 12.2], there are some nice examples including pictures. One may have a look at some of them. Another example to look at for a system that produces chaos could be the double pendulum [KJH, §5]. The double pendulum is a pendulum with another pendulum attached to its end. For simplicity, we only consider the planar version as it is done in [KJH]. We look at the equations of motion for the double pendulum [KJH, §5.1] and pictures of the chaotic phase space [KJH, §5.2].

Topic : Billiards (talk with Anna-Maria)

For this talk we will mainly follow the book called *Geometry and Billiards* by S.Tabachnikov [Tab], the book *Modern Theory of Dynamical Systems* by A.Katok and B.Hasselblatt [KH, §9.2] and the book called *Notes on Dynamical Systems* by J.Moser [Mo, §2.9].
To begin with, we describe the mathematical billiard as the motion of a billiard ball on any domain in the plane as done in [Tab, beginning of §1], [KH, 9.2]. Of course, we will draw a nice picture. In this talk, we will in particular be interested in periodic trajectories [Tab, §6] on a strictly convex and bounded domain in the plane with closed smooth boundary curve. So we start explaining the concept of *periodic trajectories* in the case of a billiard system. An interesting fact goes as follows: There are billiard tables in $\mathbb{R}^2$ where every billiard trajectory has at least three bounce points, see for instance [Tab, Figure 6.6]. One could just mention this nice result. A proof seems to be rather difficult so we would omit it for the talk. To study periodic trajectories in the billiard context, it seems to be nice to follow the approach via *Poincaré's Geometric Theorem*, cf. [Tab, §6.1] and [Mo, §2.7]. We could show a sketch of a proof of [Tab, Thm. 6.2]. This result then leads to the following theorem: *On a strictly convex billiard table there exist infinitely many distinct periodic orbits.* A reference for that could be [Mo, Thm. 2.22].

Topic : Three-Body Problem (talk with Johanna)

The three-body problem is a very famous example of a Hamiltonian system in celestial mechanics. It describes a system of three bodies that mutually act via gravitational force on each other. In contrast to the two-body problem the equations of motions can not be solved generally, but only in special cases. One could take a look at special solutions such as Lagrange's homographic solutions or Euler's colinear solutions. Further one could study the restricted three-body problem, that describes the dynamics of two heavy bodies (for example moon and earth) and a third almost massless body (for example a satellite).

Restricted three-body problem:

The main source for this talk is [FK, chapter 5] to gain some inside on the physical background one can also have a look at [Bm, chapter 3.5]. Set up the general Hamiltonian for the three-body problem and explain the restictions to the circular planar restricted three-body problem [FK, chapter 5.1]. Observe that the resulting Hamiltonian is non-autonomous. Apply the time-dependent transformation generated by the angular momentum to obtain an autonomous Hamiltonian [FK, chapter 5.2-5.3]. You can then follow [FK, chapter 5.4] to find the five Lagrange points.

Topic : Geometric Quantization (talk with Johanna)

Quantization describes the process of assigning a quantum system to a given classical system. There is no general recipe how to do this, but a mathematical approach is given by geometric quantization of symplectic manifolds. The geometric quantization procedure falls into the following three steps: prequantization, polarization, and metaplectic correction. Prequantization produces a natural Hilbert space together with a quantization procedure for observables that exactly transform Poisson brackets on the classical side into commutators on the quantum side. Nevertheless, the prequantum Hilbert space is generally understood to be "too big". To obtain the quantum Hilbert space, we reduce the number of variables from $2n$ to n. Depending on how we do this reduction, we will obtain either the position Hilbert space, the momentum Hilbert space, or the Segal–Bargmann space. This is called polarization. Metaplectic corrections are not always needed and we shall not focus on them.

Geometric quantization on Euclidean space:

During this seminar we learned a lot about classical mechanics, give a motivation why classical mechanics fails at small scales and quantum mechanics is needed. A possible source could be the first chapter of [Hall], but feel free to choose your favourite motivation. Continue with the axioms of quantum mechanics [Hall, chapter 3.6], there is no need to go into detail here. Follow [Hall, chapter 22] from there on to explain geometric quantization on Eucledean space.

Topic : Classical Field theory (talk with Johanna)
Classical field theory is essentially an infinite collection of mechanical systems (one at each point in space) and hence can be viewed as an infinite-dimensional generalization of classical mechanics. More precisely, solutions of classical mechanical systems are smooth curves $t \to \gamma(t)$ from $\mathbb{R} \to M$. In classical field theory, curves from $\mathbb{R}$ are replaced by maps from a higher-dimensional source manifold. Field theories appear everywhere in physics, examples are gauge theories where the fields are connections (as the electro-magnetic field or the fields describing W- and Z- bosons in the standard model) or general relativity where the space time metric is viewed as a field.
The main source for this talk is [CH, chapter 5]. In the first part of the talk define and explain the notion of fields, the Lagrangian, the action functional [CH, p.45-47]. Continue with deriving the Euler-Lagrange equations [CH, thm. 5.2] and show that adding a total divergence does not alter the equations of motion [CH, thm. 5.6]. At this point it would be nice to proof a version of Noethers theorem, feel free to choose between thm. 5.10 (in Eucledean space) and thm. 5.17 (on manifolds). Also explaine why these laws are conservation laws [CH, chapter 5.3]. If time allows you can in the end discuss your favorite example of a field theory from [CH, chapter 5.4].

What to do now?

Study the references and clarify questions: Ask us if you need help, more references, copies of books, . . .
Make an appointement: one or two weeks before your talk to discuss your plan of the talk and to clarify open questions

Please contact the respective organiser of the talk via mail.

References

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