Metastability in the reversible inclusion process

Alessandra Bianchi [*] Sander Dommers [†] Cristian Giardinà [‡]

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Abstract

We study the condensation regime of the finite reversible inclusion process, i.e., the inclusion process on a finite graph S with an underlying random walk that admits a reversible measure. We assume that the random walk kernel is irreducible and its reversible measure takes maximum value on a subset of vertices $S_\star \subseteq S$. We consider initial conditions corresponding to a single condensate that is localized on one of those vertices and study the metastable (or tunneling) dynamics. We find that, if the random walk restricted to $S_\star$ is irreducible, then there exists a single time-scale for the condensate motion. In this case we compute this typical time-scale and characterize the law of the (properly rescaled) limiting process. If the restriction of the random walk to $S_\star$ has several connected components, a metastability scenario with multiple time-scales emerges. We prove such a scenario, involving two additional time-scales, in a one-dimensional setting with two metastable states and nearest-neighbor jumps.

1 Introduction

The inclusion process is an interacting particle system introduced in the context of non-equilibrium statistical mechanics, as a dual process of certain diffusion processes modeling heat conduction and Fourier's law [18, 19, 20]. Besides, it is also related to models in mathematical population genetics [13], such as the Moran model, and to models of wealth distribution [15]. In addition to this, the inclusion process is interesting in its own right as an interacting particle system belonging to the class of misanthrope processes [16].

In the inclusion process, particles jump over a set S of vertices, thus the total number of particles N is conserved by the dynamics. The transitions are driven by two competing contributions to the total jump rate. Denoting by η_x the particle number at site x, and calling $r : S \times S \to \mathbb{R}_+$ the jump rates of a continuous-time irreducible random walk on S, the process is defined by the following rules (see Section 2.1 for the process generator):

*Università degli Studi di Padova, Dipartimento di Matematica, Via Trieste, 63, 35121 Padova, Italy. E-mail: bianchi@math.unipd.it

†Ruhr-Universität Bochum, Fakultät für Mathematik, Universitätsstraße 150, 44780 Bochum, Germany. E-mail: sander.dommers@ruhr-uni-bochum.de

‡Università degli Studi di Modena e Reggio Emilia, Dipartimento di Scienze Fisiche, Informatiche e Matematiche, via Campi 213/b, 41125 Modena, Italy. E-mail: cristian.giardina@unimore.it

i) firstly, particles move as continuous-time independent random walks: for a parameter $d_N > 0$, each of the η_x particles at site x waits a random time which is the minimum of exponential clocks of parameters $d_N r(x,z)$ for $z \in S$, and then jumps to site y with probability $r(x,y)/(\sum_{z\in S} r(x,z))$.

ii) secondly, particles jump because of an attractive interaction: each of the η_x particles at site x waits a random time which is the minimum of exponential clocks of parameters $\eta_z r(x,z)$ for $z \in S$, and then jumps to site y with probability $\eta_y r(x,y)/(\sum_{z\in S} \eta_z r(x,z))$.

Whereas the first contribution leads to *spreading* of the particles over the sites of S, due to the second contribution particles have a preference to *accumulate* on a few sites. This attractive interaction is in contrast to the repulsive behavior of the well-known exclusion process, where particles are subject to a hard-core potential that forbids more than one particle per site. The inclusion process is a bosonic system, whereas the exclusion process is fermionic.

The relative strength of the two contributions is tuned by the parameter sequence d_N. In the long time limit, the two contributions find a compromise into a (reversible) stationary measure that is shown explicitly in Section 2.2. As long as $d_N > 0$, this measure has mass over all the configuration space. If the sequence d_N approaches zero sufficiently fast as $N \to \infty$, then the stationary measure concentrates on a small subset of configurations. This is the phenomenon of *condensation* in the inclusion process, first studied in [21]. In the condensation regime essentially all particles accumulate on a single site of $S_\star \subseteq S$, the set over which the stationary measure of the random walk takes maximum value. The condensation phenomenon occurs in several other interacting particle systems [17], most notably the zero range process [23].

In this paper we consider the condensation regime of the inclusion process and study the dynamics of the condensate. This problem was previously considered in [22]. There, however, the authors assumed a *symmetric* random walk kernel that therefore has a uniform reversible measure on S. Here we consider instead the case of a generic reversible measure, thus allowing the possibility that $S_\star \neq S$. Depending on the properties of the underlying random walk kernel, the following *metastability* scenario with possibly *multiple time-scales* emerges from our analysis. If the restriction of the random walk kernel to $S_\star$ is still irreducible, then the system has only one time-scale. However, if such restriction is reducible into several connected components, then there exist up to three time-scales: a first time-scale over which the system moves within connected components; a second time-scale to see the jumps between components that are at graph distance equal to two; a third (even longer) time-scale for the jumps between components that are at graph distance larger than two. We point out that the origin of the multi-scale metastability can be traced back to the rates of the inclusion process and this complex scenario does not appear in the zero range process, where the rates are functions of one site only.

Our results include the characterization of the single time-scale scenario in great generality. In particular, when the system has only one time-scale, we allow any geometry and we are able to derive the rates of the limiting Markovian dynamics. We give a rigorous proof of the multiple time-scale scenario instead in the one-dimensional setting, i.e., for linear chains with nearest-neighbor jumps, whose end-points are the only maximal states of the reversible measure. In this case we fully characterize the second time-scale (together with the rates of the limiting dynamics) when $|S| = 3$, and the third time-scale when $|S| > 3$ and the stationary measure of the random walk has only two values. We conjecture that the same qualitative behavior of the motion of the condensate occurs in great generality.

The key ingredient of the proofs of our main results are *potential theory methods*. We refer in particular to the potential theoretic approach to metastability, introduced in a series of papers by Bovier, Eckhoff, Gayrard and Klein [8]–[10], and to the martingale approach, developed in some recent papers by Beltrán and Landim [3]–[6]. A general treatment of metastable systems may be found in [28], where the pathwise approach to metastability is discussed, while we refer to [11] for a recent monograph on the potential theory approach to metastability.

The paper is organized as follows. In the next Section we introduce the model and state our main results precisely. We also give an outline of the proofs. In Section 3 we analyze the metastable sets that are those configurations with all particles occupying a single site in $S_\star$. The three different time-scales and the corresponding limiting dynamics are then analyzed in Sections 4–6, respectively.

2 Model and results

2.1 Reversible inclusion process

Consider a set of sites S with $\kappa := |S| < \infty$ and let $r : S \times S \to \mathbb{R}_+$ be the jump rates of a continuous-time irreducible random walk on S, reversible with respect to some probability measure m, i.e.,

$$m(x)r(x,y) = m(y)r(y,x), \qquad \text{for all } x, y \in S. \tag{2.1}$$

Without loss of generality, we assume that $r(x,x) = 0$ for all $x \in S$.

Of special interest are the sites where m attains its maximum. Hence, we define

$$M_\star = \max\{m(x) : x \in S\}, \quad S_\star = \{x \in S : m(x) = M_\star\} \quad \text{and} \quad \kappa_\star = |S_\star|. \tag{2.2}$$

and let

$$m_\star(x) = \frac{m(x)}{M_\star}, \tag{2.3}$$

which is a normalization of m such that $m_\star(x) = 1$ if and only if $x \in S_\star$.

For a given underlying random walk we can now give the definition of the *reversible inclusion process* $\{\eta(t) : t \geq 0\}$. For each $N \geq 1$, the set of configurations E_N correspond to all the possible arrangements of N particles on S, that is

$$E_N = \{\eta \in \mathbb{N}^S : \sum_{x \in S} \eta_x = N\}. \tag{2.4}$$

The component η_x of η has to interpreted as the number of particles at site $x \in S$.

To specify the possible transitions of the dynamics, for $x, y \in S$, $x \neq y$, and $\eta \in E_N$ such that $\eta_x > 0$, let us denote by $\eta^{x,y}$ the configuration obtained from η by moving a particle from x to y:

$$(\eta^{x,y})_z = \begin{cases} \eta_x - 1, & \text{for } z = x, \\ \eta_y + 1, & \text{for } z = y, \\ \eta_z, & \text{otherwise.} \end{cases} \tag{2.5}$$

The inclusion process with N particles is then a Markov process $\{\eta(t) : t \geq 0\}$ on E_N with generator L_N, acting on functions $F : E_N \to \mathbb{R}$, given by

$$(L_N F)(\eta) = \sum_{x,y \in S} \eta_x \,(d_N + \eta_y)\, r(x,y)[F(\eta^{x,y}) - F(\eta)]\,, \tag{2.6}$$

where $\{d_N > 0 : N \in \mathbb{N}\}$ is a sequence of positive numbers that is specified later. Since we consider only finite graphs there are no restrictions on the functions F.

2.2 Condensation and metastability

The inclusion process has a stationary and reversible probability measure $\mu_N(\eta)$, given by a product measure of negative binomials conditioned to a total number of particles N, i.e.,

$$\mu_N(\eta) = \frac{1}{Z_{N,S}} \prod_{x \in S} m_\star(x)^{\eta_x} w_N(\eta_x), \tag{2.7}$$

where

$$w_N(k) = \frac{\Gamma(k + d_N)}{k!\Gamma(d_N)}, \tag{2.8}$$

and

$$Z_{N,S} = \sum_{\eta \in E_N} \prod_{x \in S} m_\star(x)^{\eta_x} w_N(\eta_x). \tag{2.9}$$

We abbreviate

$$m_\star^\eta := \prod_{x \in S} m_\star(x)^{\eta_x} \qquad \text{and} \qquad w_N(\eta) = \prod_{x \in S} w_N(\eta_x), \tag{2.10}$$

so that (2.7) becomes

$$\mu_N(\eta) = \frac{1}{Z_{N,S}} m_\star^\eta w_N(\eta). \tag{2.11}$$

The stationary measure is unique, because the underlying random walk, and hence also the inclusion process, is irreducible. It can easily be checked that the measure in (2.7) satisfies the detailed balance, and thus is the reversible measure of the process.

If the parameter d_N scales to zero fast enough in the limit $N \to \infty$, then the inclusion process shows condensation, i.e., the invariant measure concentrates on disjoint sets of configurations (that we shall call *metastable sets* or *condensates*). To formalize this idea, let

$$\mathcal{E}_N^x = \{\eta \in E_N : \eta_x = N\}\,, \qquad x \in S_\star\,. \tag{2.12}$$

Moreover, for $S_0 \subset S_\star$, define $\mathcal{E}_N(S_0) = \bigcup_{x \in S_0} \mathcal{E}_N^x$ and let $\Delta = E_N \setminus \mathcal{E}_N(S_\star)$.

The following result, proved in Section 3, shows that invariant measure asymptotically concentrates on the sets (in fact singletons) $\mathcal{E}_N^x$, $x \in S_\star$, which turn out to be the metastable sets of the process:

Proposition 2.1. *For $d_N \log N \to 0$ as $N \to \infty$, and for all $x \in S_\star$,*

$$\lim_{N \to \infty} \mu_N(\mathcal{E}_N^x) = \frac{1}{\kappa_\star}. \tag{2.13}$$

As a consequence, $\lim_{N \to \infty} \mu_N(\Delta) = 0$.

The metastability problem we address in this paper is the following. Assume the process is started from a configuration corresponding to a single condensate. Then we determine the time-scale(s) over which the condensate moves and characterize the law of the process describing the motion of the condensate.

Remark 2.2. *Notice that the metastable sets $\mathcal{E}_N^x$, $x \in S_\star$, have equal μ_N-measure. It may be worth to mention that in this situation some authors prefer to speak about* tunneling behavior *rather than metastability. However, this abuse of terminology is currently quite diffuse in the mathematical literature, and we just use the word metastability.*

2.3 Results

In order to state our findings we introduce some notation. For a set $A \subset E_N$, let τ_A denote the hitting time of A:

$$\tau_A = \inf\{t \geq 0 \ : \ \eta(t) \in A\} \ . \tag{2.14}$$

Moreover, with the identification $\mathcal{E}_N^\star \equiv \mathcal{E}_N(S^\star)$, let $\eta^{\mathcal{E}_N^\star}(t)$ denote the trace process on $\mathcal{E}_N^\star$, i.e., the process obtained from $\eta(t)$ by cutting out all time periods where the system is not in $\mathcal{E}_N^\star$. Formally, for all $t \geq 0$, $\eta^{\mathcal{E}_N^\star}(t) := \eta(S_{\mathcal{E}_N^\star}(t))$ with $S_{\mathcal{E}_N^\star}(t)$ the generalized inverse of the local time $\ell_{\mathcal{E}_N^\star}(t)$:

$$\ell_{\mathcal{E}_N^\star}(t) = \int_0^t \mathbf{1}_{\{\eta(s) \in \mathcal{E}_N^\star\}} ds \quad \text{and} \quad S_{\mathcal{E}_N^\star}(t) = \sup\{s \geq 0 \ : \ \ell_{\mathcal{E}_N^\star}(s) \leq t\} \, . \tag{2.15}$$

Notice that this is still a Markov process (we refer to [3] for a proof of this result).

Finally, let us define the process

$$X_N(t) = \psi_N(\eta^{\mathcal{E}_N^\star}(t)), \tag{2.16}$$

where $\psi_N \ : \mathcal{E}_N^\star \mapsto S_\star$ is given by

$$\psi_N(\eta) = \sum_{x \in S_\star} x \cdot \mathbf{1}_{\{\eta \in \mathcal{E}_N^x\}} \, . \tag{2.17}$$

With the above notation, and the usual convention that $\mathbb{E}_\eta(\cdot)$ denotes the expectation when the process $\eta(t)$ is started from η at time $t = 0$, we prove the following:

Theorem 2.3 (First time-scale)**.** *If $d_N \log N \to 0$ as $N \to \infty$, then, for all $x \in S_\star$,*

(i) The average time to move the condensate at x to another site of $S_\star$ is given by

$$\mathbb{E}_{\mathcal{E}_N^x}(\tau_{\mathcal{E}_N(S_\star \setminus \{x\})}) = \frac{1}{\sum_{y \in S_\star, y \neq x} r(x,y)} \frac{1}{d_N} \left(1 + o(1)\right) . \tag{2.18}$$

(ii) Assume that $X_N(0) = x$. Then, the speeded-up process $\{X_N(t/d_N) \, , \, t \geq 0\}$ converges weakly on the path space $D(\mathbb{R}_+, S_\star)$ to the Markov process $\{X(t) \, , \, t \geq 0\}$ on $S_\star$, with $X(0) = x$ and generator

$$\mathcal{L}f(y) = \sum_{z \in S_\star} r(y,z)[f(z) - f(y)] \, . \tag{2.19}$$

Furthermore, the system spends negligible time outside the metastable states, i.e., $\forall T > 0$

$$\lim_{N \to \infty} \mathbb{E}_{\mathcal{E}_N^x} \left[\int_0^T \mathbf{1}_{\{\eta(s/d_N) \in \Delta\}} ds \right] = 0. \tag{2.20}$$

Remark 2.4. *The weak convergence stated in item (ii) of Theorem 2.3 refers to the path space endowed with the Skorokhod topology. We stress the fact that from this result, together with condition* (2.20)*, one can also infer the weak convergence of the speeded-up projected process* $\{\psi_N(\eta(t/d_N)), t \geq 0\}$ *to the Markov process* $\{X(t), t \geq 0\}$ *as defined above, though with a topology on the path space, called soft topology, that is weaker than the Skorokhod one. We refer to [24] for the details.*

From Theorem 2.3, we conclude that on this first time-scale the condensate can only jump between sites in $S_\star$ that are connected in the graph induced by the underlying dynamics. In particular, if $x, y \in S_\star$ are not connected by a path in $S_\star$, then the condensate will not move from x to y on the time-scale $1/d_N$. Since the inclusion process is irreducible, we therefore expect that this movement occurs on a longer time-scale.

We formalize these ideas focusing on a specific one-dimensional setting. For an integer $\kappa \geq 2$, let

$$S = [1, \kappa] \cap \mathbb{Z}, \text{ with } \quad r(x,y) \neq 0 \text{ iff } |x-y| = 1, \quad S_\star = \{1, \kappa\}, \tag{2.21}$$

that is indeed an example of dynamics that is not irreducible when restricted to $S_\star$.

For such systems we have the following result, where we say that d_N decays *subexponentially* if, for all $\delta > 0$, $\lim_{N\to\infty} d_N e^{\delta N} = \infty$.

Theorem 2.5 (Second time-scale)**.** *Consider an underlying random walk as in* (2.21)*, with* $\kappa = 3$*. Assume that* d_N *decays subexponentially and* $d_N \log N \to 0$ *as* $N \to \infty$*. Then*

(i) The average time to move the condensate between the sites of $S_\star$ *is given by*

$$\mathbb{E}_{\mathcal{E}_N^1}(\tau_{\mathcal{E}_N^3}) = \mathbb{E}_{\mathcal{E}_N^3}(\tau_{\mathcal{E}_N^1}) = \left(\frac{1}{r(1,2)} + \frac{1}{r(3,2)}\right) \cdot (1 - m_\star(2)) \cdot \frac{N}{d_N^2}(1 + o(1)). \tag{2.22}$$

(ii) Assume that $X_N(0) = x \in S_\star$*. Then, the speeded-up process* $X_N(tN/d_N^2)$ *converges weakly on the path space* $D(\mathbb{R}_+, S_\star)$ *to the Markov process* $\{X(t), t \geq 0\}$ *on* $S_\star$*, starting at* $X(0) = x$ *and jumping back and forth between* x *and* $S_\star \setminus \{x\}$ *at rate*

$$\left(\frac{1}{r(1,2)} + \frac{1}{r(3,2)}\right)^{-1} \frac{1}{1 - m_\star(2)}. \tag{2.23}$$

Furthermore, the system spends negligible time outside the metastable states, i.e., $\forall T > 0$ *and* $x \in S_\star$*,*

$$\lim_{N\to\infty} \mathbb{E}_{\mathcal{E}_N^x}\left[\int_0^T \mathbf{1}_{\{\eta(s \cdot N/d_N^2) \in \Delta\}} ds\right] = 0. \tag{2.24}$$

As will be clear from the proof of Theorem 2.5 (see Section 5) the explanation for the presence of this second time-scale is that, in order to move the condensate between sites of $S_\star = \{1, 3\}$, the system is forced to bring particles through $S \setminus S_\star = \{2\}$. This is however an unlikely event, that slows down the motion of the condensate between sites of $S_\star$ and yields a much longer transition time-scale. In this sense, we may consider the sites of $S \setminus S_\star$ as traps for the dynamics of the system.

Following this idea, the natural further question is about the presence of possibly many time-scales related to the *length of these traps*, that is to the graph-distance between disconnected sites of $S_\star$. We answer this question in the affirmative for linear systems as those defined in (2.21) with $\kappa \geq 4$, proving that an even longer time-scale is required to move the condensate between the disconnected sites $\{1, \kappa\} = S_\star$ which are at graph-distance greater than 2. We have the following:

Theorem 2.6 (Third time-scale)**.** *Consider an underlying random walk as in* (2.21)*, with* $\kappa \geq 4$ *and* $m_\star := m_\star(2) = \ldots = m_\star(\kappa - 1)$*. Assume that* d_N *decays subexponentially and* $d_N \log N \to 0$ *as* $N \to \infty$*. Then,*

(i) The average time to move the condensate between the sites of $S_\star$ *is given by*

$$\mathbb{E}_{\mathcal{E}_N^1}(\tau_{\mathcal{E}_N^\kappa}) = \mathbb{E}_{\mathcal{E}_N^\kappa}(\tau_{\mathcal{E}_N^1}) = 3\frac{m_\star}{(1-m_\star)^2}\left(\sum_{x=2}^{\kappa-2}\frac{1}{r(x,x+1)}\right)^{-1} \cdot \frac{N^2}{d_N^3}\,(1+o(1))\,. \tag{2.25}$$

(ii) Assume that $X_N(0) = x \in S_\star$*. Then, the speeded-up process* $X_N(tN^2/d_N^3)$ *converges weakly on the path space* $D(\mathbb{R}_+, S_\star)$ *to the Markov process* $\{X(t)\,,\, t \geq 0\}$ *on* $S_\star$*, starting at* $X(0) = x$ *and jumping back and forth between* x *and* $S_\star \setminus \{x\}$ *at rate*

$$3\frac{m_\star}{(1-m_\star)^2}\left(\sum_{x=2}^{\kappa-2}\frac{1}{r(x,x+1)}\right)^{-1}\,. \tag{2.26}$$

Furthermore, the system spends negligible time outside the metastable states, i.e., $\forall T > 0$ *and* $x \in S_\star$*,*

$$\lim_{N\to\infty} \mathbb{E}_{\mathcal{E}_N^x}\left[\int_0^T \mathbf{1}_{\{\eta(s\cdot N^2/d_N^3)\in\Delta\}}ds\right] = 0\,. \tag{2.27}$$

As is shown in Section 6.1, the lower bound on the capacities, which corresponds to an upper bound on the expected crossover time, excludes the possibility of the presence of longer transition time-scales (or deeper traps). Since this bound can actually be extended to more general settings beyond the one-dimensional case, we thus conjecture that the inclusion process has always at most three time-scales for the motion of the condensate, although we cannot exclude the possibility of the presence of intermediate time-scales.

2.4 Discussion

Symmetric inclusion process. The paper [22] proves results similar to those of item (ii) of Theorem 2.3 in the case where the underlying random walk is symmetric, i.e., $r(x, y) = r(y, x)$, and under the assumption that $d_N \to 0$ and $d_N N \to \infty$ as $N \to \infty$. In this case the underlying random walk is reversible with respect to the measure $m_\star \equiv 1$, so that $S = S_\star$. In particular, all the sites of $S_\star$ belong to the same connected component and the motion of the condensate involves only the first time-scale, of order $1/d_N$. Let us mention that the results of [22] were obtained by completely different techniques, namely by a direct scaling and expansion of the generator (2.6), that was shown to converge to the generator (2.19) of the limiting Markov process.

Comparison with the zero range process. The zero range process (ZRP) is a well known interacting particle system that under suitable hypotheses displays the condensation phenomenon (see e.g. [23, 2], and reference therein). The dynamics of a condensate for the ZRP has been studied in the finite reversible case in [5] by Beltrán and Landim, as a first application of the martingale approach to metastability that was proposed by the same authors [4, 6]. The results have then been generalized to the case of a diverging number of sites [12, 1], and to the totally asymmetric case [25].

The quite complete picture of metastability obtained in the ZRP, allows for a comparison with the reversible inclusion process. We stress the following similarities. In both cases: (i) the condensate is present only on sites of $S_\star$; (ii) the metastable sets are equally probable w.r.t. the equilibrium measure, and thus they are equally stable; (iii) the energetic barriers that separate the metastable sets are at most logarithmic with the number of particles, thus yielding transition times that are at most polynomial in N.

More interesting are instead differences between the two processes: (i) the ZRP has only one relevant time-scale, at which the condensate can jump directly between any sites in $S_\star$. This is due to the fact that the rates of the scaled process on $S_\star$ are given by the capacities of the underlying random walk, that are all positive by irreducibility, thus making irreducible also the condensate dynamics; (ii) the condensate of the ZRP does not consist of N particles, but only of $N - \ell_N$ particles, for some ℓ_N such that $\ell_N \to \infty$ and $\ell_N/N \to 0$ as $N \to \infty$. It is exactly due to that small number of particles wandering around the graph, that the condensate of the ZRP is able to jump to all sites of $S_\star$ on a unique time-scale.

Multiple time-scales. Our analysis yields a metastable behavior characterized – in general – by multiple time-scales. Though we prove the existence of the second and third time-scale given in Theorems 2.5 and 2.6 only for the one-dimensional setting in (2.21), we conjecture that the same time-scales show up for general underlying dynamics and that no further time-scales can occur. In fact, we expect that the leading mechanism beyond the motion of the condensate can be reduced to a train of particles moving along single paths between metastable sets. In this sense, each path can be seen as a one-dimensional system, and the results should be proved in a similar way.

However, to formalize this idea one has first to define, for each time-scale, a new family of metastable sets obtained by merging together the metastable states that are connected on a lower time-scale (a formalization of this merging can for example be found in [4]). Then, one has to show that the reduction to one-dimensional paths is correct, or in other words, that flows of particles other than those described above, are unlikely to happen. Because of the complex geometry that may appear in general situations, this may be a rather difficult task.

For other systems with multi-scale metastable behavior see, for example, the Blume-Capel model [14, 26] and the random field Curie-Weiss model [7].

Formation of the condensate An interesting question, that still remains open in this general reversible case, concerns the time-scale characterizing the formation of the condensate. In [22] this was computed in the symmetric case, and it was shown to correspond to the same time-scale on which the condensate moves between sites. In the reversible setting, when multiple time-scales are present, the time-scale on which the condensate forms, as well as the related dynamics, will be the subject of future research.

2.5 Outline of the proof

As mentioned in the introduction, to prove our theorems we will use potential theory methods. In potential theory, crucial quantities in the case of dynamics that are reversible w.r.t. a measure μ_N, are capacities between sets. Let D_N denote the Dirichlet form associated to the generator L_N, that for $F : E_N \mapsto \mathbb{R}$, is given by

$$D_N(F) = \frac{1}{2} \sum_{x,y \in S} \sum_{\eta \in E_N} \mu_N(\eta) \eta_x (d_N + \eta_y) r(x,y) [F(\eta^{x,y}) - F(\eta)]^2. \tag{2.28}$$

For two disjoint subsets A, $B \subset E_N$, the capacity between A and B can be defined through the *Dirichlet variational principle*

$$\mathrm{Cap}_N(A,B) = \inf\{D_N(F) : F \in \mathcal{F}_N(A,B)\}. \tag{2.29}$$

where

$$\mathcal{F}_N(A,B) = \{F : F(\eta) = 1 \text{ for all } \eta \in A \text{ and } F(\eta) = 0 \text{ for all } \eta \in B\}. \tag{2.30}$$

The unique minimizer of the Dirichlet principle is the *equilibrium potential*, i.e., the harmonic function $h_{A,B}$ that solves the Dirichlet problem

$$\begin{cases} L_N h(\eta) = 0, & \text{if } \eta \notin A \cup B, \\ h(\eta) = 1, & \text{if } \eta \in A, \\ h(\eta) = 0, & \text{if } \eta \in B. \end{cases} \tag{2.31}$$

It can be easily checked that

$$h_{A,B}(\eta) = \mathbb{P}_\eta(\tau_A < \tau_B)\,. \tag{2.32}$$

As pointed out in [8]–[10], one main fact about capacities in the framework of metastability, is that they are related to the mean hitting time between sets through the formula

$$\mathbb{E}_{\nu_{A,B}}(\tau_B) = \frac{\mu_N(h_{A,B})}{\mathrm{Cap}_N(A,B)}\,, \tag{2.33}$$

where $\nu_{A,B}$ is a probability measure on A such that, for all $\eta \in A$,

$$\nu_{A,B}(\eta) = \frac{\mu_N(\eta)\mathbb{P}_\eta(\tau_B < \tau_A^+)}{\mathrm{Cap}_N(A,B)}, \tag{2.34}$$

and τ_A^+ is the return time to A, i.e.,

$$\tau_A^+ = \inf\{t > 0 \,:\, \eta(t) \in A,\, \eta(s) \neq \eta(0) \text{ for some } s \in (0,t)\}\,. \tag{2.35}$$

Notice in particular, that when A is just a singleton, as in the situations we are dealing with, the measure $\nu_{A,B}$ is just a Dirac delta over the singleton. The results stated in Theorems 2.3(i), 2.5(i) and 2.6 are based on (2.33) for $A = \mathcal{E}_N^x$ and $B = \mathcal{E}_N(S_\star \setminus \{x\})$.

Capacities also play an important rôle in [3], where potential theory ideas and martingale methods have been combined in order to prove the scaling limit of suitably speeded-up processes, as the one that we have defined in (2.16). In our setting, where metastable sets are given by singletons, the approach of [3] to prove the convergence stated in Theorems 2.3(ii) and 2.5(ii), amounts to verifying the existence of a sequence $(\theta_N,\, N\geq 1)$ of positive numbers, such that, for any $x,y\in S_\star$, $x\neq y$, the following limit exists

$$p(x,y) \;:=\; \lim_{N\to\infty} \theta_N r_N^{\mathcal{E}_\star}\left(\mathcal{E}_N^x,\mathcal{E}_N^y\right)\,, \tag{2.36}$$

where $r_N^{\mathcal{E}_\star}(\,\cdot\,,\,\cdot\,)$ are the jump rates of the trace process $\eta^{\mathcal{E}_N^\star}(t)$. The sequence (θ_N) provides the proper time-scale to be used in the scaling limit and the set of asymptotic rates $(p(x,y))_{x,y\in S_\star}$ identifies the limiting dynamics. By Lemma 6.8. in [3],

$$\begin{aligned}\mu_N(\mathcal{E}_N^x) r_N^{\mathcal{E}_\star}\left(\mathcal{E}_N^x,\mathcal{E}_N^y\right) =& \frac{1}{2}\left[\mathrm{Cap}_N\left(\mathcal{E}_N^x,\mathcal{E}_N(S_\star\setminus\{x\})\right)+\mathrm{Cap}_N\left(\mathcal{E}_N^y,\mathcal{E}_N(S_\star\setminus\{y\})\right)\right.\\ &\left.-\mathrm{Cap}_N\left(\mathcal{E}_N(\{x,y\}),\mathcal{E}_N(S_\star\setminus\{x,y\})\right)\right]\,,\end{aligned} \tag{2.37}$$

so that, once more, the main tool to prove our main results turns out to be the computation of the asymptotic capacities.

The computation of the capacities in the first time-scale is performed in Section 4, while the capacities in the second and in the third time-scale are analysed in Sections 5 and 6, respectively. In all the three cases, we first provide a lower bound by restricting the Dirichlet form to a suitable subset of E_N (or flow of configurations). We then use the obtained insights to construct an approximated equilibrium potential and deduce, via the Dirichlet principle, a matching upper bound.

In our lower bounds we repeatedly use the following lemma, which uniformly bounds (parts of) the Dirichlet form from below by the effective resistance of a linear electrical network.

Lemma 2.7. *Let $R_{i,i+1}>0, i=1,\dots,k-1$. Then, for any function $F:\{1,\dots,k\}\to\mathbb{R}$,*

$$\sum_{i=1}^{k-1} R_{i,i+1}[F(i+1)-F(i)]^2 \geq [F(k)-F(1)]^2\left(\sum_{i=1}^{k-1}\frac{1}{R_{i,i+1}}\right)^{-1}. \tag{2.38}$$

Proof. Define the function

$$g(i)=\frac{F(i)-F(1)}{F(k)-F(1)}, \tag{2.39}$$

so that $g(1)=0$ and $g(k)=1$. Then,

$$\begin{aligned}\sum_{i=1}^{k-1} R_{i,i+1}[F(i+1)-F(i)]^2 &= [F(k)-F(1)]^2\sum_{i=1}^{k-1} R_{i,i+1}[g(i+1)-g(i)]^2\\ &\geq [F(k)-F(1)]^2 \inf_{\substack{h:h(1)=0,\\ h(k)=1}} \sum_{i=1}^{k-1} R_{i,i+1}[h(i+1)-h(i)]^2\\ &= [F(k)-F(1)]^2\left(\sum_{i=1}^{k-1}\frac{1}{R_{i,i+1}}\right)^{-1},\end{aligned} \tag{2.40}$$

where the last equality follows using the series law for the effective capacity of a linear chain (see, e.g., [27]). □

3 Metastable sets

In this section we study the partition function $Z_{N,S}$ and characterize its asymptotic behavior in the limit $N \to \infty$. This result is used to prove that the configurations in $\Delta = E_N \setminus \mathcal{E}_N^\star$ are very unlikely in equilibrium and that $\mathcal{E}_N^x, x \in S_\star$ are the metastable sets (Proposition 2.1). That in turn is the main ingredient for the proof of (2.20) and (2.24) in Theorems 2.3 and 2.5, respectively.

We start analyzing the weight function $w_N(\ell)$.

Lemma 3.1. *For $d_N \log N \to 0$ as $N \to \infty$, and $0 \le k < N$,*

$$\lim_{N\to\infty} \frac{(d_N + k)w_N(k)}{d_N} = \lim_{N\to\infty} \frac{(k+1)w_N(k+1)}{d_N} = 1. \tag{3.1}$$

Proof. First note that

$$(d_N + k)w_N(k) = (d_N + k)\frac{\Gamma(k + d_N)}{k!\Gamma(d_N)} = \frac{(k+1)\Gamma(k+1+d_N)}{(k+1)!\Gamma(d_N)} = (k+1)w_N(k+1), \tag{3.2}$$

so that indeed the two limits are the same.

We rewrite

$$\frac{(k+1)w_N(k+1)}{d_N} = \frac{1}{d_N}\frac{(k+1)\Gamma(k+1+d_N)}{(k+1)!\Gamma(d_N)} = \frac{1}{\Gamma(d_N+1)}\frac{\Gamma(k+1+d_N)}{\Gamma(k+1)}. \tag{3.3}$$

Clearly,

$$\lim_{N\to\infty} \frac{1}{\Gamma(d_N+1)} = \frac{1}{\Gamma(1)} = 1, \tag{3.4}$$

and

$$\frac{\Gamma(k+1+d_N)}{\Gamma(k+1)} \ge 1. \tag{3.5}$$

The upper bound follows from Wendel's inequality [29]:

$$\frac{\Gamma(k+1+d_N)}{\Gamma(k+1)} \le (k+1)^{d_N} \le N^{d_N} = e^{d_N \log N}, \tag{3.6}$$

which indeed converges to 1 by our assumption on d_N. □

We can now compute the limiting behavior of the partition function:

Proposition 3.2. *For $d_N \log N \to 0$ as $N \to \infty$,*

$$\lim_{N\to\infty} \frac{N}{d_N} Z_{N,S} = \kappa_\star. \tag{3.7}$$

Proof. Since $Z_{N,S}$ includes the $\kappa_\star$ configurations where all particles are on one of the sites in $S_\star$, it is clear that

$$Z_{N,S} \ge \kappa_\star w_N(N) = \kappa_\star \frac{d_N}{N}(1 + o(1)), \tag{3.8}$$

by Lemma 3.1.

To prove the upper bound we proceed by induction. We label the sites by $1,\dots,\kappa$, and let $E_{n,k}$ be the set of configurations with n particles on the first k sites, i.e., $E_{n,k}=E_{n,\{1,\dots,k\}}$. Let us define, for $1\le n\le N$,

$$Z_{n,k}=\sum_{\eta\in E_{n,k}} m_\star^\eta w_N(\eta). \tag{3.9}$$

By induction over k, we aim to prove that, for all $1\le n\le N$, $1\le k\le\kappa$, and N large enough,

$$Z_{n,k}\le \frac{d_N(1+o(1))}{n}\sum_{s=1}^{k} m_\star(s)^n + C_k\frac{d_N^2\log n}{n}, \tag{3.10}$$

where $C_k<\infty$ is a constant that only depends on k and may change from line to line.

We start the induction with $k=1$, for which clearly, by Lemma 3.1,

$$Z_{n,1}=m_\star(1)^n w_N(n)=\frac{d_N(1+o(1))}{n}m_\star(1)^n\,. \tag{3.11}$$

Assume that (3.10) holds true for $k-1$ and for all $1\le n\le N$. Then, using the induction hypothesis and Lemma 3.1,

$$\begin{aligned}
Z_{n,k}&=m_\star(k)^n w_N(n)+Z_{n,k-1}+\sum_{\ell=1}^{n-1} m_\star(k)^\ell w_N(\ell)Z_{n-\ell,k-1}\\
&\le \frac{d_N(1+o(1))}{n}\left(m_\star(k)^n+\sum_{s=1}^{k-1} m_\star(s)^n\right)+C_{k-1}\frac{d_N^2\log n}{n}\\
&\quad+\sum_{\ell=1}^{n-1} m_\star(k)^\ell\frac{d_N(1+o(1))}{\ell}\left(\left(\sum_{s=1}^{k-1} m_\star(s)^{n-\ell}\right)\frac{d_N(1+o(1))}{(n-\ell)}+C_{k-1}\frac{d_N^2\log(n-\ell)}{n-\ell}\right).
\end{aligned} \tag{3.12}$$

Using that $m_\star(k)\le 1$ and $d_N\log(n-\ell)=o(1)$ by assumption, and for N large enough, the sum in ℓ can be bounded from above by

$$C_k d_N^2\sum_{\ell=1}^{n-1}\frac{1}{\ell(n-\ell)}=2C_k d_N^2\sum_{\ell=1}^{n/2}\frac{1}{\ell(n-\ell)}=2C_k\frac{d_N^2}{n}\sum_{\ell=1}^{n/2}\frac{1}{\ell(1-\ell/n)}\,. \tag{3.13}$$

Since $\ell\le n/2$ we have that $(1-\ell/n)\ge\frac12$. Hence, we can bound (3.13) from above by

$$4C_k\frac{d_N^2}{n}\sum_{\ell=1}^{n/2}\frac{1}{\ell}\le 4C_k\frac{d_N^2\log n}{n}. \tag{3.14}$$

This proves the induction step. Thus,

$$\begin{aligned}
Z_{N,\kappa}&\le\frac{d_N(1+o(1))}{N}\sum_{s=1}^{\kappa} m_\star(s)^N+C_\kappa\frac{d_N^2\log N}{N} \qquad (3.15)\\
&=\kappa_\star\frac{d_N(1+o(1))}{N}+\frac{d_N(1+o(1))}{N}\left(\left(\sum_{s\notin S_\star} m_\star(s)^N\right)+C_\kappa d_N\log N\right)\\
&=\kappa_\star\frac{d_N}{N}(1+o(1)). \qquad (3.16)
\end{aligned}$$

The proposition follows by combining this upper bound with the lower bound in (3.8). □

Combining these results, Proposition 2.1 follows trivially:

Proof of Proposition 2.1. For all $x \in S_\star$, by Lemma 3.1 and Proposition 3.2,

$$\lim_{N\to\infty} \mu_N(\mathcal{E}_N^x) = \lim_{N\to\infty} \frac{w_N(N)}{Z_{N,S}} = \lim_{N\to\infty} \frac{N \cdot w_N(N)}{d_N} \frac{d_N}{N \cdot Z_{N,S}} = \frac{1}{\kappa^*}\,. \tag{3.17}$$

As a consequence,

$$\lim_{N\to\infty} \mu_N(\Delta) = 1 - \sum_{x\in S_\star} \lim_{N\to\infty} \mu_N(\mathcal{E}_N^x) = 0\,. \tag{3.18}$$

□

4 Dynamics of the condensate on the first time-scale

In this section we analyze capacities on the time-scale $1/d_N$ and prove Theorem 2.3. We prove the lower bound on capacities in Section 4.1, the upper bound in Section 4.2, and we give the proof of Theorem 2.3 in Section 4.3.

4.1 Lower bound on capacities

Proposition 4.1. *For a nonempty subset $S_\star^1 \subsetneq S_\star$, let $S_\star^2 = S_\star \setminus S_\star^1$. Then, for $d_N \log N \to 0$ as $N \to \infty$,*

$$\liminf_{N\to\infty} \frac{1}{d_N} \mathrm{Cap}_N\left(\mathcal{E}_N(S_\star^1), \mathcal{E}_N(S_\star^2)\right) \geq \frac{1}{\kappa_\star} \sum_{x\in S_\star^1} \sum_{y\in S_\star^2} r(x,y). \tag{4.1}$$

Proof. Let

$$A_N^{x,y} = \{\eta \in E_N \,:\, \eta_x + \eta_y = N\}\,. \tag{4.2}$$

Fix a function $F \in \mathcal{F}_N(\mathcal{E}_N(S_\star^1), \mathcal{E}_N(S_\star^2))$. Then,

$$\begin{aligned} D_N(F) &= \frac{1}{2} \sum_{x,y\in S} \sum_{\eta\in E_N} \mu_N(\eta)\eta_x\,(d_N + \eta_y)\, r(x,y)[F(\eta^{x,y}) - F(\eta)]^2 \\ &\geq \frac{1}{2} \sum_{x\in S_\star^1} \sum_{y\in S_\star^2} \sum_{\eta\in A_N^{x,y}} \mu_N(\eta)\eta_x\,(d_N + \eta_y)\, r(x,y)[F(\eta^{x,y}) - F(\eta)]^2 \\ &\qquad + \frac{1}{2} \sum_{y\in S_\star^2} \sum_{x\in S_\star^1} \sum_{\eta\in A_N^{x,y}} \mu_N(\eta)\eta_y\,(d_N + \eta_x)\, r(y,x)[F(\eta^{y,x}) - F(\eta)]^2 \\ &= \sum_{x\in S_\star^1} \sum_{y\in S_\star^2} r(x,y) \sum_{\eta\in A_N^{x,y}} \mu_N(\eta)\eta_x\,(d_N + \eta_y)\,[F(\eta^{x,y}) - F(\eta)]^2, \end{aligned} \tag{4.3}$$

by reversibility. Note that the set $A_N^{x,y}$ can be parametrized by the number of particles at x, and is thus a one-dimensional set. For a fixed couple $x, y \in S_\star$, let G be the restriction of F to the set

$A_N^{x,y}$, i.e., for $\eta \in A_N^{x,y}$ such that $\eta_x = \ell$, define $G(\ell) := F(\eta)$. Then we can rewrite

$$\sum_{\eta \in A_N^{x,y}} \mu_N(\eta)\eta_x (d_N + \eta_y) [F(\eta^{x,y}) - F(\eta)]^2$$
$$= \sum_{\ell=1}^{N} \frac{w_N(\ell) w_N(N-\ell)}{Z_{N,S}} \ell (d_N + N - \ell) [G(\ell - 1) - G(\ell)]^2, \tag{4.4}$$

where we used that $m_\star(x) = m_\star(y) = 1$.

Using Lemma 3.1 for all $1 \le \ell \le N$,

$$w_N(\ell) w_N(N-\ell)\ell (d_N + N - \ell) = d_N^2 (1 + o(1)), \tag{4.5}$$

so that (4.4) equals

$$\frac{d_N^2(1 + o(1))}{Z_{N,S}} \sum_{\ell=1}^{N} [G(\ell - 1) - G(\ell)]^2. \tag{4.6}$$

Note that $G(0) = 0$ and $G(N) = 1$, so that it follows from Lemma 2.7 that

$$\sum_{\ell=1}^{N} [G(\ell - 1) - G(\ell)]^2 \ge \frac{1}{N}. \tag{4.7}$$

Hence,

$$D_N(F) \ge \frac{d_N^2}{N Z_{N,S}} \sum_{x \in S_\star^1} \sum_{y \in S_\star^2} r(x, y), \tag{4.8}$$

and the proposition follows from Proposition 3.2. □

4.2 Upper bound on capacities

Proposition 4.2. *For a nonempty subset $S_\star^1 \subsetneq S_\star$, let $S_\star^2 = S_\star \setminus S_\star^1$. Then, for $d_N \log N \to 0$ as $N \to \infty$,*

$$\limsup_{N \to \infty} \frac{1}{d_N} \mathrm{Cap}_N \left(\mathcal{E}_N(S_\star^1), \mathcal{E}_N(S_\star^2) \right) \le \frac{1}{\kappa_\star} \sum_{x \in S_\star^1} \sum_{y \in S_\star^2} r(x, y). \tag{4.9}$$

The strategy of the proof is to provide a suitable test function $F \in \mathcal{F}_N(\mathcal{E}(S_\star^1), \mathcal{E}(S_\star^2))$ to plug in the Dirichlet principle

$$\mathrm{Cap}_N(\mathcal{E}(S_\star^1), \mathcal{E}(S_\star^2)) = \inf\{D_N(F) : F \in \mathcal{F}_N(\mathcal{E}(S_\star^1), \mathcal{E}(S_\star^2))\}. \tag{4.10}$$

We first describe how we construct the test function, and then study the corresponding Dirichlet form by splitting it into several parts, and analyzing each of them separately. We conclude the section collecting all the results and providing the proof of the above proposition.

Construction of the test function. Inspired by the lower bound derived in the previous section, we want $F(\eta)$ to be approximately equal to $G^*(\eta_x)$, where G^* is the minimizer of $\sum_{\ell=1}^N [G(\ell-1)-G(\ell)]^2$, which is given by

$$G^*(\ell) = \frac{\ell}{N}. \tag{4.11}$$

To avoid difficulties for small and large values of η_x, we choose an arbitrary small $\varepsilon > 0$ and set the function equal to 0 if $\eta_x/N \le \varepsilon$, and equal to 1 if $\eta_x/N \ge 1-\varepsilon$.

For values $\eta_x/N \in (\varepsilon, 1-\varepsilon)$, we approximate $G^*(\eta_x)$ with a smooth function $\phi_\varepsilon(\eta_x/N)$ defined as in [5]. That is, $\phi_\varepsilon : [0,1] \to [0,1]$ is a smooth nondecreasing function satisfying $\phi_\varepsilon(t) + \phi_\varepsilon(1-t) = 1$ for all $t \in [0,1]$, $\phi_\varepsilon(t) = 0$ for $t \le \varepsilon$, $\phi_\varepsilon(t) = 1$ for $t \ge 1-\varepsilon$, and $\phi'_\varepsilon(t) \le 1+\sqrt{\varepsilon}$ for all $t \in [0,1]$. Such a function exists since $(1+\sqrt{\varepsilon})$ times the length of the interval $[\varepsilon, 1-\varepsilon]$ is strictly bigger than 1 for ε small enough.

All together, for any $x \in S$, we define the functions $F_x : E_N \mapsto \mathbb{R}$ as

$$F_x(\eta) = \phi_\varepsilon(\eta_x/N), \tag{4.12}$$

and similarly, for $S^1 \subset S$, the functions $F_{S^1} : E_N \mapsto \mathbb{R}$ as

$$F_{S^1}(\eta) = \sum_{x\in S^1} F_x(\eta). \tag{4.13}$$

Split of the Dirichlet form. To split the Dirichlet form, define, for a set $A \subseteq E_N$,

$$D_N(F, A) = \frac{1}{2}\sum_{\eta\in A} \mu_N(\eta) \sum_{z,w\in S} \eta_z(d_N+\eta_w) r(z,w)[F(\eta^{z,w}) - F(\eta)]^2. \tag{4.14}$$

Also define

$$A_N^x = \bigcup_{y\in S\setminus\{x\}} A_N^{x,y}. \tag{4.15}$$

We can then write

$$D_N(F_{S^1}) = D_N(F_{S^1}, E_N) = D_N(F_{S^1}, \bigcup_{x\in S^1} A_N^x) + D_N(F_{S^1}, E_N \setminus \bigcup_{x\in S^1} A_N^x). \tag{4.16}$$

By definition

$$D_N(F_{S^1}, \bigcup_{x\in S^1} A_N^x) = D_N(F_{S^1}, \bigcup_{x\in S^1}\bigcup_{y\in S\setminus\{x\}} A_N^{x,y}). \tag{4.17}$$

If $\{x_1, y_1\} \ne \{x_2, y_2\}$ and $\eta \in A_N^{x_1,y_1} \cap A_N^{x_2,y_2}$, then either $x_1 = x_2$ and $\eta_{x_1} = N$, or $y_1 = y_2$ and $\eta_{y_1} = N$. In both cases, $F_{S^1}(\eta^{z,w}) = F_{S^1}(\eta)$ for all $z, w \in S$ because of the definition of ϕ_ε. Therefore, we can write

$$D_N(F_{S^1}, \bigcup_{x\in S^1} A_N^x) = \sum_{x\in S^1}\sum_{y\in S^2} D_N(F_{S^1}, A_N^{x,y}) + \frac{1}{2}\sum_{x,y\in S^1} D_N(F_{S^1}, A_N^{x,y}), \tag{4.18}$$

where $S^2 = S \setminus S^1$.

Dirichlet form inside tubes. The main contribution to the Dirichlet form comes from configurations inside tubes between sites $x \in S^1, y \in S^2$, as the next lemma shows.

Lemma 4.3. *Let $x \in S^1$ and $y \in S^2$. Then, for $d_N \log N \to 0$ as $N \to \infty$,*

$$\lim_{\varepsilon \to 0} \limsup_{N \to \infty} \frac{1}{d_N} D_N(F_{S^1}, A_N^{x,y}) \leq \frac{1}{\kappa_\star} r(x,y) \mathbb{1}\{x, y \in S_\star\}. \tag{4.19}$$

Proof. Note that if $\eta \in A_N^{x,y}$, then for $v \in S^1 \setminus \{x\}$ we have that $F_v(\eta) = F_v(\eta^{z,w}) = 0$, since $\eta_v, \eta_v^{z,w} \leq 1 < \varepsilon N$. Hence,

$$D_N(F_{S^1}, A_N^{x,y}) = D_N(F_x, A_N^{x,y}). \tag{4.20}$$

Note also that for configurations such that $\eta_x < \varepsilon N$, or $\eta_x > (1-\varepsilon)N$, we have that $F_x(\eta^{z,w}) = F_x(\eta)$. Hence, we can restrict the sum to configurations η such that $\varepsilon N \leq \eta_x \leq (1-\varepsilon)N$ and get

$$D_N(F_x, A_N^{x,y}) = \frac{1}{2Z_{N,S}} \sum_{j=\varepsilon N}^{(1-\varepsilon)N} m_\star(x)^j m_\star(y)^{N-j} w_N(j) w_N(N-j) \sum_{z,w \in S} \eta_z (d_N + \eta_w) r(z,w) [F_x(\eta^{z,w}) - F_x(\eta)]^2. \tag{4.21}$$

Since $F_x(\eta) = \phi_\varepsilon(\eta_x)$ does not change if the number of particles on x stays the same, we can further rewrite this, also using reversibility, as

$$\begin{aligned} D_N(F_x, A_N^{x,y}) = \frac{1}{Z_{N,S}} \sum_{j=\varepsilon N}^{(1-\varepsilon)N} & m_\star(x)^j m_\star(y)^{N-j} w_N(j) w_N(N-j) \\ & \Big\{ j(d_N + N - j) r(x,y) \Big[\phi_\varepsilon\big(\tfrac{j-1}{N}\big) - \phi_\varepsilon\big(\tfrac{j}{N}\big)\Big]^2 \\ & + \sum_{z \in S \setminus \{x,y\}} j d_N r(x,z) \Big[\phi_\varepsilon\big(\tfrac{j-1}{N}\big) - \phi_\varepsilon\big(\tfrac{j}{N}\big)\Big]^2 \Big\}. \end{aligned} \tag{4.22}$$

Because of the bound on $\phi_\varepsilon'(t)$, we have that

$$\Big|\phi_\varepsilon\big(\tfrac{j+1}{N}\big) - \phi_\varepsilon\big(\tfrac{j}{N}\big)\Big| \leq \frac{1+\sqrt{\varepsilon}}{N}. \tag{4.23}$$

Thus, also using Lemma 3.1,

$$\begin{aligned} D_N(F_x, A_N^{x,y}) &\leq \frac{d_N^2(1+o(1))}{N^2 Z_{N,S}} (1+\sqrt{\varepsilon})^2 m_\star(x)^{\varepsilon N} m_\star(y)^{\varepsilon N} \sum_{j=\varepsilon N}^{(1-\varepsilon)N} \Big\{ r(x,y) + \sum_{z \in S \setminus \{x,y\}} \frac{d_N}{N-j} r(x,z) \Big\} \\ &= \frac{d_N(1+o(1))}{\kappa_\star} (1+\sqrt{\varepsilon})^2 (1-2\varepsilon) m_\star(x)^{\varepsilon N} m_\star(y)^{\varepsilon N} r(x,y), \end{aligned} \tag{4.24}$$

where in the second equality we used Proposition 3.2. Hence,

$$\limsup_{N \to \infty} \frac{1}{d_N} D_N(F_x, A_N^{x,y}) \leq \frac{r(x,y)}{\kappa_\star} (1+\sqrt{\varepsilon})^2 (1-2\varepsilon) \mathbb{1}\{x, y \in S_\star\}, \tag{4.25}$$

and the lemma follows by taking the limit $\varepsilon \to 0$. □

The contribution to the Dirichlet form coming from configurations inside a tube between sites $x, y \in S_1$ is negligible, as the next lemma shows.

Lemma 4.4. *Let $x, y \in S^1$. Then, for $d_N \log N \to 0$ as $N \to \infty$,*

$$\lim_{N\to\infty} \frac{1}{d_N} D_N(F_{S^1}, A_N^{x,y}) = 0. \tag{4.26}$$

Proof. Again, note that if $\eta \in A_N^{x,y}$, then for $v \in S^1 \setminus \{x, y\}$ we have that $F_v(\eta) = F_v(\eta^{z,w}) = 0$, since $\eta_v, \eta_v^{z,w} \le 1 < \varepsilon N$. Thus,

$$D_N(F_{S^1}, A_N^{x,y}) = D_N(F_x + F_y, A_N^{x,y}). \tag{4.27}$$

If a particle moves from x to y, or viceversa, the total number of particles on sites x and y stays equal to N and hence

$$F_x(\eta) + F_y(\eta) = F_x(\eta^{x,y}) + F_y(\eta^{x,y}) = F_x(\eta^{y,x}) + F_y(\eta^{y,x}) = 1, \tag{4.28}$$

since by definition, $\phi_\varepsilon(x) + \phi_\varepsilon(1-x) = 1$ for all $x \in [0,1]$. We can use again that F_v does not change if the number of particles on v stays the same, and restrict the sum to configurations with $\varepsilon N \le \eta_x \le (1-\varepsilon)N$. Thus

$$\begin{aligned} D_N(F_{S^1}, A_N^{x,y}) &= \frac{1}{2} \sum_{\eta \in A_N^{x,y}} \mu_N(\eta) \sum_{z \in S \setminus \{x,y\}} \Big\{ \eta_x d_N r(x,z) \Big[F_x(\eta^{x,z}) - F_x(\eta) \Big]^2 \\ &\qquad + \eta_y d_N r(y,z) \Big[F_y(\eta^{y,z}) - F_y(\eta) \Big]^2 \Big\} \\ &= \frac{d_N}{2Z_{N,S}} \sum_{j=\varepsilon N}^{(1-\varepsilon)N} m_\star(x)^j m_\star(y)^{N-j} w_N(j) w_N(N-j) \Big\{ j r(x,z) \Big[\phi_\varepsilon\big(\frac{j-1}{n}\big) - \phi_\varepsilon\big(\frac{j}{n}\big) \Big]^2 \\ &\qquad + (N-j) r(y,z) \Big[\phi_\varepsilon\big(\frac{N-j-1}{N}\big) - \phi_\varepsilon\big(\frac{N-j}{N}\big) \Big]^2 \Big\}. \end{aligned} \tag{4.29}$$

Using Lemma 3.1, (4.23) and $m_\star(x), m_\star(y) \le 1$, we can bound

$$\begin{aligned} D_N(F_{S^1}, A_N^{x,y}) &\le \frac{d_N^3(1+o(1))}{2N^2 Z_{N,S}} (1+\sqrt{\varepsilon})^2 \sum_{j=\varepsilon N}^{(1-\varepsilon)N} \left\{ \frac{r(x,z)}{N-j} + \frac{r(y,z)}{j} \right\} \\ &= \frac{d_N(1+o(1))}{2\kappa_\star} (1+\sqrt{\varepsilon})^2 (1-2\varepsilon) o(1). \end{aligned} \tag{4.30}$$

Then finally,

$$\lim_{N\to\infty} \frac{1}{d_N} D_N(F_{S^1}, A_N^{x,y}) = \lim_{N\to\infty} d_N o(1) = 0. \tag{4.31}$$

$\square$

Dirichlet form outside tubes We finally show in the next lemma, that the configurations outside the collections of tubes A_N^z gives a negligible contribution to the Dirichlet form.

Lemma 4.5. *For $d_N \log N \to 0$ as $N \to \infty$,*

$$\lim_{N\to\infty} \frac{1}{d_N} D_N(F_{S^1}, E_N \setminus \bigcup_{x\in S^1} A_N^x) = 0. \tag{4.32}$$

Proof. As in [5], by the Cauchy-Schwarz inequality we get

$$[F_{S^1}(\eta^{z,w}) - F_{S^1}(\eta)]^2 = \left[\sum_{x\in S^1}[F_x(\eta^{z,w}) - F_x(\eta)]\right]^2 \le |S^1| \sum_{x\in S^1}[F_x(\eta^{z,w}) - F_x(\eta)]^2, \tag{4.33}$$

and then

$$D_N(F_{S^1}, E_N \setminus \bigcup_{z\in S^1} A_N^z) \le |S^1| \sum_{x\in S^1} D_N(F_x, E_N \setminus \bigcup_{z\in S^1} A_N^z) \le |S^1| \sum_{x\in S^1} D_N(F_x, E_N \setminus A_N^x). \tag{4.34}$$

Again, we can restrict the sum to configurations with $\varepsilon N \le \eta_x \le (1-\varepsilon)N$. Furthermore, if $\eta \in E_N \setminus A_N^x$ and $\eta_x = j$, all sites besides x have at most $N-j-1$ particles, and thus

$$D_N(F_x, E_N \setminus A_N^x) = \frac{1}{2} \sum_{j=\varepsilon N}^{(1-\varepsilon)N} \sum_{\substack{\eta:\eta_x=j \\ \forall y\neq x:\, \eta_y\le N-j-1}} \mu_N(\eta) \sum_{z,w\in S} \eta_z(d_N+\eta_w) r(z,w)[F_x(\eta^{z,w}) - F_x(\eta)]^2. \tag{4.35}$$

Note that if $z, w \neq x$, then $F_x(\eta^{z,w}) = F_x(\eta)$, since F_x only depends on the number of particles on x. Hence,

$$\begin{aligned} D_N(F_x, E_N \setminus A_N^x) &= \frac{1}{2} \sum_{j=\varepsilon N}^{(1-\varepsilon)N} \sum_{\substack{\eta\in E_N:\eta_x=j \\ \forall y\neq x:\, \eta_y\le N-j-1}} \mu_N(\eta) \\ &\quad \sum_{y\in S\setminus\{x\}} \Big\{ \eta_x(d_N+\eta_y) r(x,y)[F_x(\eta^{x,y}) - F_x(\eta)]^2 + \eta_y(d_N+\eta_x) r(y,x)[F_x(\eta^{y,x}) - F_x(\eta)]^2 \Big\} \\ &= \frac{1}{2Z_{N,S}} \sum_{j=\varepsilon N}^{(1-\varepsilon)N} m_\star(x)^j w_N(j) \sum_{\substack{\eta\in E_N:\eta_x=j \\ \forall y\neq x:\, \eta_y\le N-j-1}} \prod_{y\in S\setminus\{x\}} \Big(m_\star(y)^{\eta_y} w_N(\eta_y) \Big) \\ &\quad \sum_{y\in S\setminus\{x\}} \Big\{ j(d_N+\eta_y) r(x,y) \Big[\phi_\varepsilon\big(\tfrac{j-1}{N}\big) - \phi_\varepsilon\big(\tfrac{j}{N}\big)\Big]^2 + \eta_y(d_N+j) r(y,x) \Big[\phi_\varepsilon\big(\tfrac{j+1}{N}\big) - \phi_\varepsilon\big(\tfrac{j}{N}\big)\Big]^2 \Big\}. \end{aligned} \tag{4.36}$$

Since $|S| < \infty$, we can bound $r(x,y), r(y,x) \le \max_{z,w\in S} r(z,w) =: R$ and $m_\star(x) \le 1$, and also bound $\max\{j(d_N+N-j), (N-j)(d_N+j)\} \le j(N-j)(1+o(1))$. Combining this with (4.23), we get

$$\begin{aligned} &D_N(F_x, E_N \setminus A_N^x) \\ &\le R(\kappa-1)\frac{(1+\sqrt{\varepsilon})^2}{N^2 Z_{N,S}} \sum_{j=\varepsilon N}^{(1-\varepsilon)N} w_N(j) j(N-j)(1+o(1)) \sum_{\substack{\eta\in E_N:\eta_x=j \\ \forall y\neq x:\, \eta_y\le N-j-1}} \prod_{y\in S\setminus\{x\}} \Big(m_\star(y)^{\eta_y} w_N(\eta_y) \Big). \end{aligned} \tag{4.37}$$

Notice that the last sum can be written as

$$Z_{N-j,S\setminus\{x\}} - \sum_{y\in S\setminus\{x\}} m_\star(y)^{N-j} w_N(N-j) \le \frac{d_N}{N-j} o(1), \tag{4.38}$$

where the inequality follows from (3.10). Hence, also using Lemma 3.1 and Proposition 3.2,

$$\begin{aligned} D_N(F_x, E_N \setminus A_N^x) &\le R(\kappa-1)\frac{(1+\sqrt{\varepsilon})^2}{N^2 Z_{N,S}}(1-2\varepsilon)N d_N^2(1+o(1))o(1) \\ &= R\frac{(\kappa-1)}{\kappa_\star}(1+\sqrt{\varepsilon})^2(1-2\varepsilon)d_N(1+o(1))o(1), \end{aligned} \tag{4.39}$$

from which it follows that

$$\frac{1}{d_N} D_N(F_x, E_N \setminus A_N^x) = o(1), \tag{4.40}$$

and that together with (4.34) proves the lemma. □

Combining these lemmas, we can now prove Proposition 4.2.

Proof of Proposition 4.2. Let $S^1 \subsetneq S$ be such that $S_\star^1 \subseteq S^1$ and $S_\star^2 \subseteq S \setminus S^1 =: S^2$. Note that if $\eta \in \mathcal{E}_N(S_\star^2)$ then $F_{S_1}(\eta) = 0$, and if $\eta \in \mathcal{E}_N(S_\star^1)$ then $F_{S_1}(\eta) = 1$. Hence, $F_{S_1} \in \mathcal{F}_N(\mathcal{E}(S_\star^1), \mathcal{E}(S_\star^2))$. Therefore, by (4.10),

$$\mathrm{Cap}_N(\mathcal{E}(S_\star^1), \mathcal{E}(S_\star^2)) \le D_N(F_{S^1}). \tag{4.41}$$

We can split the right hand side according to (4.16) and (4.18), and the proposition then follows from Lemmas 4.3, 4.4 and 4.5. □

4.3 Proof of Theorem 2.3

Proof of Theorem 2.3(i). As a consequence of Propositions 4.1 and 4.2, we have that for nonempty subsets $S_\star^1 \subsetneq S_\star$ and $S_\star^2 = S_\star \setminus S_\star^1$, and $d_N \log N \to 0$ as $N \to \infty$,

$$\lim_{N\to\infty} \frac{1}{d_N} \mathrm{Cap}_N\left(\mathcal{E}_N(S_\star^1), \mathcal{E}_N(S_\star^2)\right) = \frac{1}{\kappa_\star} \sum_{x\in S_\star^1} \sum_{y\in S_\star^2} r(x,y). \tag{4.42}$$

In view of (2.32) and (2.33), in order to prove the statement (i) we need to provide an asymptotic formula for the μ_N-average of the equilibrium potential $\mathbb{P}_\eta\left(\tau_{\mathcal{E}_N^x} < \tau_{\mathcal{E}_N(S_\star\setminus\{x\})}\right)$. Since this is trivially equal to 1 for $\eta \in \mathcal{E}_N^x$, and equal to 0 for $\eta \in \mathcal{E}_N(S_\star \setminus \{x\})$, we have on one hand

$$\sum_{\eta\in E_N} \mu_N(\eta) \cdot \mathbb{P}_\eta\left(\tau_{\mathcal{E}_N^x} < \tau_{\mathcal{E}_N(S_\star\setminus\{x\})}\right) \ge \mu_N(\mathcal{E}_N^x), \tag{4.43}$$

and on the other hand

$$\sum_{\eta\in E_N} \mu_N(\eta) \cdot \mathbb{P}_\eta\left(\tau_{\mathcal{E}_N^x} < \tau_{\mathcal{E}_N(S_\star\setminus x)}\right) \le \sum_{\substack{\eta\in E_N \\ \eta\notin\mathcal{E}_N(S_\star\setminus\{x\})}} \mu_N(\eta) = \mu_N(\mathcal{E}_N^x) + \mu_N(\Delta). \tag{4.44}$$

From these bounds and Proposition 2.1, it follows

$$\sum_{\eta\in E_N} \mu_N(\eta)\cdot \mathbb{P}_\eta\left(\tau_{\mathcal{E}_N^x} < \tau_{\mathcal{E}_N(S_\star\setminus\{x\})}\right) = \frac{1}{\kappa_\star}(1+o(1))\,, \tag{4.45}$$

that together with (4.42) concludes the proof of (2.18). □

Proof of Theorem 2.3(ii). We stress once more that in our setting, where metastable sets are just singletons, the convergence of the speeded-up process follows from Theorem 2.7 of [3] once the condition (2.36) of Section 2.5 (called condition **(H0)** in [3]) is verified for the sequence $\theta_N = 1/d_N$, $N \geq 1$.

By Lemma 6.8 of [3], that we have recalled in (2.37), and using Proposition 2.1 and (4.42), we get that for any $x, y \in S_\star$, $x \neq y$,

$$\lim_{N\to\infty} \frac{1}{d_N} r_N^{\mathcal{E}_\star}\left(\mathcal{E}_N^x, \mathcal{E}_N^y\right) = r(x,y)\,. \tag{4.46}$$

To prove (2.20) observe that by the stationarity of μ_N we have

$$\begin{aligned}\mathbb{E}_{\mathcal{E}_N^x}\left[\int_0^T \mathbb{1}\{\eta(s/d_N)\in\Delta\}\,\mathrm{d}s\right] &\leq \frac{1}{\mu_N(\mathcal{E}_N^x)}\sum_{\eta\in E_N}\mu_N(\eta)\mathbb{E}_\eta\left[\int_0^T \mathbb{1}\{\eta(s/d_N)\in\Delta\}\,\mathrm{d}s\right]\\ &= T\cdot\frac{\mu_N(\Delta)}{\mu_N(\mathcal{E}_N^x)}\,.\end{aligned} \tag{4.47}$$

Then (2.20) follows from Proposition 2.1. This concludes the proof of theorem.

□

5 Dynamics of the condensate on the second time-scale

This section is organized similarly to the previous one. We first provide a lower bound on capacities, then a matching upper bound, and finally we give the proof of Theorem 2.5.

5.1 Lower bound on capacities

Proposition 5.1. *Let the underlying random walk be as in* (2.21)*, with* $\kappa = 3$*. Then, for* $d_N \log_N \to 0$ *as* $N\to\infty$*,*

$$\liminf_{N\to\infty} \frac{N}{d_N^2}\mathrm{Cap}_N\left(\mathcal{E}_N(1),\mathcal{E}_N(3)\right) \geq \left(\frac{1}{r(1,2)}+\frac{1}{r(3,2)}\right)^{-1}\frac{1}{1-m_\star(2)}. \tag{5.1}$$

Proof. Fix a function $F\in\mathcal{F}(\mathcal{E}_N(1),\mathcal{E}_N(3))$. Using reversibility, we can write the Dirichlet form of F as

$$\begin{aligned}D_N(F) =&\sum_{\eta\in E_N}\mu_N(\eta)\bigg(\eta_1(d_N+\eta_2)r(1,2)\left[F(\eta^{1,2})-F(\eta)\right]^2+\eta_2(d_N+\eta_3)r(2,3)\left[F(\eta^{2,3})-F(\eta)\right]^2\bigg)\\ =&\sum_{\xi\in E_{N-1}}\bigg(\mu_N(\xi+\partial_1)(\xi_1+1)(d_N+\xi_2)r(1,2)\left[F(\xi+\partial_2)-F(\xi+\partial_1)\right]^2\\ &\qquad+\mu_N(\xi+\partial_2)(\eta_2+1)(d_N+\xi_3)r(2,3)\left[F(\xi+\partial_3)-F(\xi+\partial_2)\right]^2\bigg),\end{aligned} \tag{5.2}$$

where $\xi + \partial_z$ denotes a configuration ξ with $N-1$ particles, and with one extra particle on z.

For some fixed L and N big enough, we can restrict the Dirichlet form of F by only considering configurations $\xi \in E_{N-1}$ such that $\xi_1 = j$, $\xi_2 = \ell$ and $\xi_3 = N - j - \ell - 1$, with $\ell \le L$. On this set of configurations, we then define the function $G(j,\ell,z) := F(\xi + \partial_z)$. With abuse of notation, we also define $G(N-\ell,\ell,3) := F(\eta)$ where $\eta_1 = N-\ell$, $\eta_2 = \ell$ and $\eta_3 = 0$. We can then write

$$\begin{aligned} D_N(F) \ge \frac{1}{Z_{N,S}} \sum_{\ell=0}^{L} \sum_{j=0}^{N-\ell-1} \\ &\Big\{ w_N(j+1) m_\star(2)^\ell w_N(\ell) w_N(N-j-\ell-1)(j+1)(d_N+\ell) r(1,2)[G(j,\ell,2)-G(j,\ell,1)]^2 \\ &\quad + w_N(j) m_\star(2)^{\ell+1} w_N(\ell+1) w_N(N-j-\ell-1)(\ell+1)(d_N+N-j-\ell-1) \\ &\quad \cdot r(2,3)[G(j,\ell,3)-G(j,\ell,2)]^2 \Big\}, \end{aligned} \tag{5.3}$$

where we used that $m_\star(2) r(2,3) = r(3,2)$ by the reversibility of the underlying random walk. From inequality (4.5), we can then bound (5.3) from below by

$$\begin{aligned} \frac{d_N^2}{Z_{N,S}} \sum_{\ell=0}^{L} m_\star(2)^\ell \sum_{j=0}^{N-\ell-1} &\Big\{ w_N(N-j-\ell-1) r(1,2)[G(j,\ell,2)-G(j,\ell,1)]^2 \\ &+ w_N(j) r(3,2)[G(j,\ell,3)-G(j,\ell,2)]^2 \Big\}. \end{aligned} \tag{5.4}$$

Moreover, let us define

$$\tilde{w}_N(j) = \begin{cases} d_N, & \text{if } j = 0, \\ w_N(j), & \text{if } j > 0, \end{cases} \tag{5.5}$$

so that $w_N(0) = 1 = \tilde{w}_N(0) + (1-d_N)$ and hence

$$\begin{aligned} D_N(F) \ge \frac{d_N^2}{Z_{N,S}} \sum_{\ell=0}^{L} m_\star(2)^\ell \sum_{j=0}^{N-\ell-1} &\Big\{ \tilde{w}_N(N-j-\ell-1) r(1,2)[G(j,\ell,2)-G(j,\ell,1)]^2 \\ &+ \tilde{w}_N(j) r(3,2)[G(j,\ell,3)-G(j,\ell,2)]^2 \Big\} \\ + (1-d_N) \frac{d_N^2}{Z_{N,S}} \sum_{\ell=0}^{L-1} m_\star(2)^\ell &\Big(r(1,2)[G(N-\ell-1,\ell,2)-G(N-\ell-1,\ell,1)]^2 \\ &+ r(3,2)[G(0,\ell,3)-G(0,\ell,2)]^2 \Big). \end{aligned} \tag{5.6}$$

Using Lemma 2.7 with

$$g(z) = \frac{G(j,\ell,z)-G(j,\ell,3)}{G(j,\ell,1)-G(j,\ell,3)}, \tag{5.7}$$

we can bound

$$\begin{aligned} &\tilde{w}_N(N-j-\ell-1) r(1,2)[G(j,\ell,2)-G(j,\ell,1)]^2 + \tilde{w}_N(j) r(3,2)[G(j,\ell,3)-G(j,\ell,2)]^2 \\ &\quad \ge [G(j,\ell,1)-G(j,\ell,3)]^2 \left(\frac{1}{\tilde{w}_N(N-j-\ell-1) r(1,2)} + \frac{1}{\tilde{w}_N(j) r(3,2)} \right)^{-1}. \end{aligned} \tag{5.8}$$

Observing that $G(j,\ell,1) = G(j+1,\ell,3)$, and using Lemma 2.7 again, we bound

$$\sum_{j=0}^{N-\ell-1} [G(j,\ell,1) - G(j,\ell,3)]^2 \left(\frac{1}{\tilde{w}_N(N-j-\ell-1) r(1,2)} + \frac{1}{\tilde{w}_N(j) r(3,2)} \right)^{-1} \tag{5.9}$$

$$\geq [G(N-\ell,\ell,3) - G(0,\ell,3)]^2 \left(\sum_{j=0}^{N-\ell-1} \left(\frac{1}{\tilde{w}_N(N-j-\ell-1) r(1,2)} + \frac{1}{\tilde{w}_N(j) r(3,2)} \right) \right)^{-1}.$$

By reversing the summing order of the first term, the sum over j equals

$$\left(\frac{1}{r(1,2)} + \frac{1}{r(3,2)} \right) \sum_{j=0}^{N-\ell-1} \frac{1}{\tilde{w}_N(j)} = \left(\frac{1}{r(1,2)} + \frac{1}{r(3,2)} \right) \frac{1}{d_N} \left(1 + \sum_{j=1}^{N-\ell-1} j(1+o(1)) \right)$$

$$= \left(\frac{1}{r(1,2)} + \frac{1}{r(3,2)} \right) \frac{N^2}{2d_N}(1+o(1)), \tag{5.10}$$

since $\ell = o(N)$. Hence,

$$\begin{aligned}
D_N(F) \geq{}& \frac{d_N^2}{Z_{N,S}} \sum_{\ell=0}^{L} m_\star(2)^\ell [G(N-\ell,\ell,3) - G(0,\ell,3)]^2 \left(\left(\frac{1}{r(1,2)} + \frac{1}{r(3,2)} \right) \frac{N^2}{2d_N} \right)^{-1} (1+o(1)) \\
&+ (1-d_N) \frac{d_N^2}{Z_{N,S}} \sum_{\ell=0}^{L-1} m_\star(2)^\ell \Big(r(1,2)[G(N-\ell-1,\ell,2) - G(N-\ell-1,\ell,1)]^2 \\
&\qquad + r(3,2)[G(0,\ell,3) - G(0,\ell,2)]^2 \Big) \\
={}& \frac{d_N^2}{Z_{N,S}} \sum_{\ell=0}^{L} \Bigg\{ m_\star(2)^\ell \frac{2d_N}{N^2} \left(\frac{1}{r(1,2)} + \frac{1}{r(3,2)} \right)^{-1} (1+o(1)) [G(N-\ell,\ell,3) - G(0,\ell,3)]^2 \\
&+ (1-d_N) \sum_{p=0}^{\ell-1} \frac{m_\star(2)^p}{L-p} \Big(r(1,2)[G(N-p-1,p+1,3) - G(N-p,p,3)]^2 \\
&\qquad + r(3,2)[G(0,p,3) - G(0,p+1,3)]^2 \Big) \Bigg\},
\end{aligned} \tag{5.11}$$

where we used that for every function $f(p)$, we have the identity

$$\sum_{\ell=0}^{L-1} f(\ell) = \sum_{\ell=0}^{L} \sum_{p=0}^{\ell-1} \frac{f(p)}{L-p}. \tag{5.12}$$

Using Lemma 2.7, we can bound

$$\begin{aligned}
&\sum_{p=0}^{\ell-1} \frac{m_\star(2)^p}{L-p} [G(N-p-1,p+1,3) - G(N-p,p,3)]^2 \\
&\qquad \geq [G(N-\ell,\ell,3) - G(N,0,3)]^2 \left(\sum_{p=0}^{\ell-1} \frac{L-p}{m_\star(2)^p} \right)^{-1} \\
&\qquad \geq [G(N-\ell,\ell,3) - G(N,0,3)]^2 \frac{m_\star(2)^\ell}{L^2},
\end{aligned} \tag{5.13}$$

and
$$\sum_{p=0}^{\ell-1}\frac{m_\star(2)^p}{L-p}[G(0,p,3)-G(0,p+1,3)]^2\geq[G(0,0,3)-G(0,\ell,3)]^2\frac{m_\star(2)^\ell}{L^2}. \tag{5.14}$$

Thus,

$$\begin{aligned}D_N(F)\geq\frac{d_N^2}{Z_{N,S}}\sum_{\ell=0}^{L}m_\star(2)^\ell\Big\{&\frac{2d_N}{N^2}\left(\frac{1}{r(1,2)}+\frac{1}{r(3,2)}\right)^{-1}(1+o(1))[G(N-\ell,\ell,3)-G(0,\ell,3)]^2\\&+r(1,2)\frac{1-d_N}{L^2}[G(N-\ell,\ell,3)-G(N,0,3)]^2+r(3,2)\frac{1-d_N}{L^2}[G(0,0,3)-G(0,\ell,3)]^2\Big\},\end{aligned} \tag{5.15}$$

and since $G(0,0,3)=0$ and $G(N,0,3)=1$, we get

$$\begin{aligned}D_N(F)&\geq\frac{d_N^2}{Z_{N,S}}\sum_{\ell=0}^{L}m_\star(2)^\ell\left\{\frac{N^2}{2d_N}\left(\frac{1}{r(1,2)}+\frac{1}{r(3,2)}\right)(1+o(1))+\frac{r(1,2)L^2}{1-d_N}+\frac{r(3,2)L^2}{1-d_N}\right\}^{-1}\\&=\frac{d_N^2}{N}\frac{2d_N}{NZ_{N,S}}\left(\frac{1}{r(1,2)}+\frac{1}{r(3,2)}\right)^{-1}(1+o(1))\sum_{\ell=0}^{L}m_\star(2)^\ell\,.\end{aligned} \tag{5.16}$$

By Proposition 3.2, $\lim_{N\to\infty}\frac{d_N}{NZ_{N,S}}=\frac{1}{\kappa_\star}=\frac{1}{2}$. Hence,

$$\liminf_{N\to\infty}\frac{N}{d_N^2}D_N(F)\geq\left(\frac{1}{r(1,2)}+\frac{1}{r(3,2)}\right)^{-1}\sum_{\ell=0}^{L}m_\star(2)^\ell, \tag{5.17}$$

and the proposition follows by taking $L\to\infty$. □

Remark 5.2. *The above lemma can be generalized to systems with arbitrary set S and underlying dynamics, such that $S_\star=\{x,y\}$ with x,y sites at graph-distance 2. In that case, we have the lower bound*

$$\liminf_{N\to\infty}\frac{N}{d_N^2}\mathrm{Cap}_N\left(\mathcal{E}_N(x),\mathcal{E}_N(y)\right)\geq\sum_{v\in S\setminus\{x,y\}}\left(\frac{1}{r(x,v)}+\frac{1}{r(y,v)}\right)^{-1}\frac{1}{1-m_\star(v)}. \tag{5.18}$$

This can easily be proved by restricting the Dirichlet form to those jumps that have at most one vertex $v\in S\setminus S_\star$ with a positive number of particles, and then proceeding as above. Notice that if it does not exists $v\in S$ such that $r(x,v)>0$ and $r(y,v)>0$ then the r.h.s. of (5.18) is zero, suggesting the existence of an additional (larger) time-scale.

5.2 Upper bound on capacities

Proposition 5.3. *Let the underlying random walk be as in (2.21), with $\kappa=3$. Furthermore, suppose that d_N decays subexponentially and $d_N\log N\to 0$ as $N\to\infty$. Then,*

$$\limsup_{N\to\infty}\frac{N}{d_N^2}\mathrm{Cap}_N\left(\mathcal{E}_N(1),\mathcal{E}_N(3)\right)\leq\left(\frac{1}{r(1,2)}+\frac{1}{r(3,2)}\right)^{-1}\frac{1}{1-m_\star(2)}. \tag{5.19}$$

Proof. Since there are only three sites, the space E_N is parametrized by the number of particles on 1 and 2. We want to choose a test function so that (5.8) and (5.9) approximately hold with equality. That is, when a particle jumps from 1 to 2 we want the contribution to be approximately a constant times

$$\frac{1}{\tilde{w}_N(N-j-\ell-1)r(1,2)} \approx \frac{N-j-\ell-1}{d_N r(1,2)} \approx \frac{N}{d_N}\frac{1}{r(1,2)}(1-\frac{j}{N}), \tag{5.20}$$

and when a particle jumps from 3 to 2 it should be approximately the same constant times

$$\frac{1}{\tilde{w}_N(j)r(3,2)} \approx \frac{1}{r(3,2)}\frac{j}{d_N} = \frac{N}{d_N}\frac{1}{r(3,2)}\frac{j}{N}. \tag{5.21}$$

Hence, with ϕ_ε defined as in Section 4.2, we let

$$G(j,\ell) = 2\left(\frac{1}{r(1,2)}+\frac{1}{r(3,2)}\right)^{-1}\left(\frac{1}{r(1,2)}\int_0^{\phi_{2\varepsilon}\left(\frac{j-1}{N}\right)}(1-x)\,\mathrm{d}x + \frac{1}{r(3,2)}\int_0^{\phi_{2\varepsilon}\left(\frac{j-1}{N}+(\frac{\ell}{N}\wedge\varepsilon)\right)} x\,\mathrm{d}x\right), \tag{5.22}$$

and consider the test function

$$F(\eta) = G(\eta_1,\eta_2). \tag{5.23}$$

The constant is chosen such that $G(N,0)=1$ and $G(0,0)=0$, so that indeed $F \in \mathcal{F}_N(\mathcal{E}_N(1),\mathcal{E}_N(3))$, while the extremes of the integrals are chosen in such a way that the first integral only contributes when a particle jumps between 1 and 2, and the second integral only when the jump is between 2 and 3, with a truncation so that ℓ cannot be too big.

Using reversibility we can write the Dirichlet form of F as

$$\begin{aligned}
D_N(F) &= \sum_{\eta\in E_N}\mu_N(\eta)\Big(\eta_1(d_N+\eta_2)r(1,2)\left[F(\eta^{1,2})-F(\eta)\right]^2 + \eta_2(d_N+\eta_3)r(2,3)\left[F(\eta^{2,3})-F(\eta)\right]^2\Big)\\
&= \sum_{\xi\in E_{N-1}}\Big(\mu_N(\xi+\partial_1)(\xi_1+1)(d_N+\xi_2)r(1,2)\left[F(\xi+\partial_2)-F(\xi+\partial_1)\right]^2\\
&\qquad + \mu_N(\xi+\partial_2)(\eta_2+1)(d_N+\xi_3)r(2,3)\left[F(\xi+\partial_3)-F(\xi+\partial_2)\right]^2\Big)\\
&= \frac{1}{Z_{N,S}}\sum_{\ell=0}^{N-1}\sum_{j=0}^{N-\ell-1}\Big(w_N(j+1)m_\star(2)^\ell w_N(\ell)w_N(N-j-\ell-1)(j+1)(d_N+\ell)r(1,2)\\
&\qquad\qquad [G(j,\ell+1)-G(j+1,\ell)]^2\\
&\qquad + w_N(j)m_\star(2)^{\ell+1}w_N(\ell+1)w_N(N-j-\ell-1)(\ell+1)(d_N+N-j-\ell-1)r(2,3)\\
&\qquad\qquad [G(j,\ell)-G(j,\ell+1)]^2\Big).
\end{aligned} \tag{5.24}$$

By the definition of G, we can compute

$$\begin{aligned}
&G(j+1,\ell)-G(j,\ell+1) \\
&\quad = 2\left(\frac{1}{r(1,2)}+\frac{1}{r(3,2)}\right)^{-1}\left(\frac{1}{r(1,2)}\int_{\phi_{2\varepsilon}\left(\frac{j-1}{N}\right)}^{\phi_{2\varepsilon}\left(\frac{j}{N}\right)}(1-x)\,\mathrm{d}x + \frac{1}{r(3,2)}\int_{\phi_{2\varepsilon}\left(\frac{j-1}{N}+(\frac{\ell+1}{N}\wedge\varepsilon)\right)}^{\phi_{2\varepsilon}\left(\frac{j}{N}+(\frac{\ell}{N}\wedge\varepsilon)\right)} x\,\mathrm{d}x\right),
\end{aligned} \tag{5.25}$$

which is 0 for $j \le \varepsilon N$, and $j > (1-2\varepsilon)N$. Also

$$G(j,\ell+1) - G(j,\ell) = 2\left(\frac{1}{r(1,2)} + \frac{1}{r(3,2)}\right)^{-1} \frac{1}{r(3,2)} \int_{\phi_{2\varepsilon}\left(\frac{j-1}{N}+(\frac{\ell}{N}\wedge\varepsilon)\right)}^{\phi_{2\varepsilon}\left(\frac{j-1}{N}+(\frac{\ell+1}{N}\wedge\varepsilon)\right)} x \,\mathrm{d}x, \tag{5.26}$$

which is 0 for $\ell \ge \varepsilon N$, and also for $j \le \varepsilon N$ or $j > (1-2\varepsilon)N$. Hence, by Lemma 3.1,

$$\begin{aligned} D_N(F) = {} & \frac{d_N^3(1+o(1))}{Z_{N,S}} \sum_{\ell=0}^{\varepsilon N-1} m_\star(2)^\ell \Bigg(\sum_{j=\varepsilon N}^{(1-2\varepsilon)N} \frac{1}{N-j-\ell-1} r(1,2)\left[G(j,\ell+1) - G(j+1,\ell)\right]^2 \\ & + \sum_{j=\varepsilon N}^{(1-2\varepsilon)N} \frac{1}{j} r(3,2)\left[G(j,\ell) - G(j,\ell+1)\right]^2 \Bigg) \\ & + \frac{d_N^2(1+o(1))}{Z_{N,S}} \sum_{\ell=\varepsilon N}^{N-1} m_\star(2)^\ell \sum_{j=\varepsilon N}^{N-\ell-1} w_N(N-j-\ell-1) r(1,2)\left[G(j,\ell+1) - G(j+1,\ell)\right]^2, \end{aligned} \tag{5.27}$$

where we also used the reversibility of the underlying random walk to substitute $m_\star(2)r(2,3) = r(3,2)$.

For $\ell \le \varepsilon N - 1$, the second integral in (5.25) is 0, so that

$$\begin{aligned} & \sum_{j=\varepsilon N}^{(1-2\varepsilon)N} \frac{1}{N-j-\ell-1} r(1,2)\left[G(j,\ell+1) - G(j+1,\ell)\right]^2 \\ & = \sum_{j=\varepsilon N}^{(1-2\varepsilon)N} \frac{1}{N-j-\ell-1} r(1,2)\left[2\left(\frac{1}{r(1,2)} + \frac{1}{r(3,2)}\right)^{-1} \frac{1}{r(1,2)} \int_{\phi_{2\varepsilon}\left(\frac{j-1}{N}\right)}^{\phi_{2\varepsilon}\left(\frac{j}{N}\right)} (1-x)\,\mathrm{d}x\right]^2 \\ & = 4\left(\frac{1}{r(1,2)} + \frac{1}{r(3,2)}\right)^{-2} \frac{1}{r(1,2)} \sum_{j=\varepsilon N}^{(1-2\varepsilon)N} \int_{\phi_{2\varepsilon}\left(\frac{j-1}{N}\right)}^{\phi_{2\varepsilon}\left(\frac{j}{N}\right)} (1-x)\,\mathrm{d}x \frac{1}{N} \int_{\phi_{2\varepsilon}\left(\frac{j-1}{N}\right)}^{\phi_{2\varepsilon}\left(\frac{j}{N}\right)} \frac{1-x}{1-\frac{j+\ell+1}{N}}\,\mathrm{d}x. \end{aligned} \tag{5.28}$$

Then, by the properties of $\phi_{2\varepsilon}$ and using that $\frac{\ell+2}{N} \le 2\varepsilon$ for N big enough, we get

$$\int_{\phi_{2\varepsilon}\left(\frac{j-1}{N}\right)}^{\phi_{2\varepsilon}\left(\frac{j}{N}\right)} \frac{1-x}{1-\frac{j+\ell+1}{N}}\,\mathrm{d}x \le \left(\phi_{2\varepsilon}\left(\tfrac{j}{N}\right) - \phi_{2\varepsilon}\left(\tfrac{j-1}{N}\right)\right) \frac{1-\phi_{2\varepsilon}\left(\frac{j-1}{N}\right)}{1-\frac{j+\ell+1}{N}} \le \frac{1+\sqrt{\varepsilon}}{N} \frac{\phi_{2\varepsilon}\left(1-\frac{j-1}{N}\right)}{1-\frac{j-1}{N}-2\varepsilon}. \tag{5.29}$$

Using the fundamental theorem of calculus, and that $\phi_{2\varepsilon}(2\varepsilon) = 0$,

$$\phi_{2\varepsilon}\left(1-\tfrac{j-1}{N}\right) = \int_{2\varepsilon}^{1-\frac{j-1}{N}} \phi_{2\varepsilon}'(x)\,\mathrm{d}x \le \left(1 - \frac{j-1}{N} - 2\varepsilon\right)(1+\sqrt{\varepsilon}). \tag{5.30}$$

Hence,

$$\int_{\phi_{2\varepsilon}\left(\frac{j-1}{N}\right)}^{\phi_{2\varepsilon}\left(\frac{j}{N}\right)} \frac{1-x}{1-\frac{j+\ell+1}{N}}\,\mathrm{d}x \le \frac{1}{N}(1+\sqrt{\varepsilon})^2, \tag{5.31}$$

so that

$$\sum_{j=\varepsilon N}^{(1-2\varepsilon)N} \frac{1}{N-j-\ell-1} r(1,2)\left[G(j,\ell+1)-G(j+1,\ell)\right]^2$$

$$\leq \frac{4(1+\sqrt{\varepsilon})^2}{N^2}\left(\frac{1}{r(1,2)}+\frac{1}{r(3,2)}\right)^{-2}\frac{1}{r(1,2)}\sum_{j=\varepsilon N}^{(1-2\varepsilon)N}\int_{\phi_{2\varepsilon}\left(\frac{j-1}{N}\right)}^{\phi_{2\varepsilon}\left(\frac{j}{N}\right)}(1-x)\,\mathrm{d}x$$

$$= \frac{2(1+\sqrt{\varepsilon})^2}{N^2}\left(\frac{1}{r(1,2)}+\frac{1}{r(3,2)}\right)^{-2}\frac{1}{r(1,2)}. \tag{5.32}$$

Similarly, we can use (5.26) to bound, for $\ell \leq \varepsilon N-1$,

$$\sum_{j=\varepsilon N}^{(1-2\varepsilon)N} \frac{1}{j} r(3,2)\left[G(j,\ell)-G(j,\ell+1)\right]^2$$

$$= 4\left(\frac{1}{r(1,2)}+\frac{1}{r(3,2)}\right)^{-2}\frac{1}{r(3,2)}\sum_{j=\varepsilon N}^{(1-2\varepsilon)N}\int_{\phi_{2\varepsilon}\left(\frac{j+\ell-1}{N}\right)}^{\phi_{2\varepsilon}\left(\frac{j+\ell}{N}\right)}x\,\mathrm{d}x\frac{1}{N}\int_{\phi_{2\varepsilon}\left(\frac{j+\ell-1}{N}\right)}^{\phi_{2\varepsilon}\left(\frac{j+\ell}{N}\right)}\frac{x}{j/N}\,\mathrm{d}x$$

$$\leq \frac{2(1+\sqrt{\varepsilon})^2}{N^2}\left(\frac{1}{r(1,2)}+\frac{1}{r(3,2)}\right)^{-2}\frac{1}{r(3,2)}. \tag{5.33}$$

To bound the third line of (5.27), notice that $|G(j,\ell+1)-G(j+1,\ell)| \leq 1$ and $m_\star(2)^\ell \leq m_\star(2)^{\varepsilon N}$, so that

$$\sum_{\ell=\varepsilon N}^{N-1} m_\star(2)^\ell \sum_{j=\varepsilon N}^{N-\ell-1} w_N(N-j-\ell-1)\left[G(j,\ell+1)-G(j+1,\ell)\right]^2 \tag{5.34}$$

$$\leq m_\star(2)^{\varepsilon N}\sum_{\ell=\varepsilon N}^{N-1}\left(1+\sum_{j=\varepsilon N}^{N-\ell-2}\frac{d_N(1+o(1))}{N-j-\ell-1}\right) \leq m_\star(2)^{\varepsilon N} N(1+d_N\log N(1+o(1))).$$

Hence, we obtain

$$D_N(F) \leq \frac{d_N^3(1+o(1))}{N^2 Z_{N,S}} 2(1+\sqrt{\varepsilon})^2\left(\frac{1}{r(1,2)}+\frac{1}{r(3,2)}\right)^{-1}\sum_{\ell=0}^{\varepsilon N-1} m_\star(2)^\ell$$
$$+\frac{d_N^2(1+o(1))}{Z_{N,S}} m_\star(2)^{\varepsilon N} N(1+d_N\log N). \tag{5.35}$$

Taking the limit $N\to\infty$ gives

$$\limsup_{N\to\infty}\frac{N}{d_N^2}D_N(F) \leq \lim_{N\to\infty}\frac{d_N(1+o(1))}{NZ_{N,S}}2(1+\sqrt{\varepsilon})^2\left(\frac{1}{r(1,2)}+\frac{1}{r(3,2)}\right)^{-1}\sum_{\ell=0}^{\varepsilon N} m_\star(2)^\ell$$
$$+\frac{d_N(1+o(1))}{NZ_{N,S}}\frac{1}{d_N} m_\star(2)^{\varepsilon N} N^2(1+d_N\log N)$$
$$= \frac{2(1+\sqrt{\varepsilon})^2}{\kappa_\star}\left(\frac{1}{r(1,2)}+\frac{1}{r(3,2)}\right)^{-1}\frac{1}{1-m_\star(2)}, \tag{5.36}$$

where we used that d_N decays subexponentially to show that the second part converges to 0. The proposition follows by taking the limit $\varepsilon\to 0$ and noting that $\kappa_\star = 2$. $\square$

5.3 Proof of Theorem 2.5

The proof runs similarly to that of Theorem 2.3.

Proof of Theorem 2.5(i). As a consequence of Propositions 5.1 and 5.3, if d_N decays subexponentially and $d_N \log N \to 0$ as $N \to \infty$,

$$\lim_{N\to\infty} \frac{N}{d_N^2} \mathrm{Cap}_N\left(\mathcal{E}_N(1), \mathcal{E}_N(3)\right) = \left(\frac{1}{r(1,2)} + \frac{1}{r(3,2)}\right)^{-1} \frac{1}{1-m_\star(2)}\,. \tag{5.37}$$

In view of (2.33), recalling that $\mathrm{Cap}_N(A,B) = \mathrm{Cap}_N(B,A)$, and applying (4.45), this provides formula (2.22). □

Proof of Theorem 2.5(ii). As in the proof of Theorem 2.3(ii), the convergence follows from Theorem 2.7 of [3] once condition (2.36) of Section 2.5 is verified for the sequence $\theta_N = N/d_N^2$, $N \geq 1$. By Lemma 6.8 of [3] (see (2.37) in Section 2.5) and using Proposition 2.1 and (5.37), we get

$$\lim_{N\to\infty} \frac{N}{d_N^2} r_N^{\mathcal{E}_\star}\left(\mathcal{E}_N^1, \mathcal{E}_N^3\right) = \lim_{N\to\infty} \frac{N}{d_N^2} r_N^{\mathcal{E}_\star}\left(\mathcal{E}_N^3, \mathcal{E}_N^1\right) = \left(\frac{1}{r(1,2)} + \frac{1}{r(3,2)}\right)^{-1} \frac{1}{1-m_\star(2)}\,, \tag{5.38}$$

proving (2.36). Finally, (2.24) is proved similarly to (2.20). □

6 Dynamics of the condensate on the third time-scale

In this last section we study the third time-scale that appears when the condensate moves between sites in $S_\star$ that are at graph-distance larger than 2.

6.1 Lower bound on capacities

Proposition 6.1. *Let the underlying random walk be as in* (2.21)*, with* $\kappa \geq 4$ *and*

$$m_\star := \min_{x=2,\ldots,\kappa-1} m_\star(x). \tag{6.1}$$

Then, for $d_N \log_N \to 0$ *as* $N \to \infty$*,*

$$\liminf_{N\to\infty} \frac{N^2}{d_N^3} \mathrm{Cap}_N\left(\mathcal{E}_N(1), \mathcal{E}_N(\kappa)\right) \geq 3\frac{m_\star}{(1-m_\star)^2}\left(\sum_{x=2}^{\kappa-2} \frac{1}{r(x,x+1)}\right)^{-1}. \tag{6.2}$$

Proof. This lower bound is given by transporting particles from 1 to κ, in such a way that transport involves at most two neighboring sites of $S \setminus S_*$. To see this, consider any function $F : E_N \mapsto \mathbb{R}$ such that $F(\eta_1 = N) = 1$ and $F(\eta_\kappa = N) = 0$. We first use reversibility to write

$$\begin{aligned} D_N(F) &= \frac{1}{2} \sum_{\eta\in E_N} \mu_N(\eta) \sum_{z,w\in S} \eta_z(d_N + \eta_w) r(z,w)[F(\eta^{z,w}) - F(\eta)]^2 \\ &= \sum_{\eta\in E_N} \mu_N(\eta) \sum_{x=1}^{\kappa-1} \eta_x(d_N + \eta_{x+1}) r(x,x+1)[F(\eta^{x,x+1}) - F(\eta)]^2. \end{aligned} \tag{6.3}$$

We bound this from below by considering only configurations η parametrized by (j, x, ℓ, k), such that $\eta_y = 0$ for all $y \notin \{1, x, x+1, \kappa\}$, $\eta_1 = j \geq 0$, $\eta_x = k$ and $\eta_{x+1} = \ell - k$ (meaning that $\eta_x + \eta_{x+1} = \ell$), with x a site from 1 to $\kappa - 1$, $k \leq \ell$ and $\ell \leq L$ with L fixed, and $\eta_\kappa = N - j - \ell$. We write, with abuse of notation, $\mu_N(j, x, \ell, k)$ and $F(j, x, \ell, k)$ for $\mu_N(\eta)$ and $F(\eta)$, respectively.

This means that, for a given ℓ, particles will be transported from 1 to κ in groups of ℓ at a time. To make sure we do not leave out too many configurations, we also choose a number of particles $0 \leq i < \ell$, which will be transported first one at a time. When these i particles and as many groups of ℓ particles as possible have been transported, there might be some particles left, since $N - i$ might not be divisible by ℓ, which are also transported one at a time in the end.

We have to make sure that we are not double counting contributions of particle jumps that occur for different ℓ and i. For this, note that if a particle jumps from $2 \leq x \leq \kappa - 2$ to $x + 1$, ℓ and i are uniquely determined. It turns out that these are also the only main contributions. For jumps that can occur multiple times, for example the first particle jump will occur for all ℓ and i, we divide these contributions by L^2, which is the maximum number of times this contribution can occur. Since these contributions are negligible, as we show below, this does not change the result.

We can hence write

$$D_N(F) \geq \sum_{\ell=1}^{L} \sum_{i=0}^{\ell-1} \left(A_1(\ell, i) + A_2(\ell, i) + A_3(\ell, i) \right), \tag{6.4}$$

where $A_1(\ell, i)$ is the contribution of moving the first i particles, $A_3(\ell, i)$ of moving the last $N - (i + (\lfloor N/\ell \rfloor - 1)\ell)$ particles and $A_2(\ell, i)$ of moving the particles in groups of ℓ. We focus on this last term first.

Let $0 \leq n < \lfloor N/\ell \rfloor - 1$ be the number of groups that has already been transported. Then we can split $A_2(\ell, i)$ further as

$$A_2(\ell, i) = \sum_{n=0}^{\lfloor N/\ell \rfloor - 2} A_{2,n}(\ell, i), \tag{6.5}$$

where

$$\begin{aligned} A_{2,n}(\ell, i) = \frac{1}{L^2} \sum_{k=1}^{\ell} & \mu_N(N - (n+1)\ell - i + k, 2, \ell - k, 0)(N - (n+1)\ell - i + k)(d_N + \ell - k) r(1, 2) \\ & \times \left[F(N - (n+1)\ell - i + k - 1, 2, \ell - k + 1, 0) - F(N - (n+1)\ell - i + k, 2, \ell - k, 0) \right]^2 \\ + & \sum_{x=2}^{\kappa-2} \sum_{k=1}^{\ell} \mu_N(N - (n+1)\ell - i, x, \ell, k) k (d_N + \ell - k) r(x, x+1) \\ & \times \left[F(N - (n+1)\ell - i, x, \ell, k - 1) - F(N - (n+1)\ell - i, x, \ell, k) \right]^2 \\ + & \frac{1}{L^2} \sum_{k=1}^{\ell} \mu_N(N - (n+1)\ell - i, \kappa - 2, k, 0) k (d_N + n\ell + i + \ell - k) r(\kappa - 1, \kappa) \\ & \times \left[F(N - (n+1)\ell - i, \kappa - 1, k - 1, 0) - F(N - (n+1)\ell - i, \kappa - 1, k, 0) \right]^2. \end{aligned} \tag{6.6}$$

Using Lemma 3.1 and that $m_\star(x) \geq m_\star$ for all $x = 2, \dots, \kappa - 1$ we can bound this as

$$\begin{aligned}
A_{2,n}(\ell, i) \geq \frac{d_N^4(1+o(1))}{Z_N} \bigg\{ & \frac{1}{d_N L^2} \frac{r(1,2)}{n\ell + i} \sum_{k=1}^{\ell} m_\star^{\ell-k} \\
& \times [F(N - (n+1)\ell - i + k - 1, 2, \ell - k + 1, 0) - F(N - (n+1)\ell - i + k, 2, \ell - k, 0)]^2 \\
& + \frac{m_\star^\ell}{(N - (n+1)\ell - i)(n\ell + i)} \sum_{x=2}^{\kappa-2} r(x, x+1) \sum_{k=1}^{\ell} 1 \\
& \times [F(N - (n+1)\ell - i, x, \ell, k-1) - F(N - (n+1)\ell - i, x, \ell, k)]^2 \\
& + \frac{1}{d_N L^2} \frac{r(\kappa - 1, \kappa)}{N - (n+1)\ell - i} \sum_{k=1}^{\ell} m_\star^k \\
& \times [F(N - (n+1)\ell - i, \kappa - 1, k - 1, 0) - F(N - (n+1)\ell - i, \kappa - 1, k, 0)]^2 \bigg\}. \qquad (6.7)
\end{aligned}$$

Using Lemma 2.7 twice, we can bound this further as

$$\begin{aligned}
A_{2,n}(\ell, i) \geq \frac{d_N^4(1+o(1))}{Z_N} \bigg\{ & \frac{1}{d_N L^2} \frac{r(1,2)}{n\ell + i} \left(\sum_{k=1}^{\ell} \frac{1}{m_\star^{\ell-k}} \right)^{-1} \\
& \times [F(N - (n+1)\ell - i, 2, \ell, 0) - F(N - n\ell - i, 2, 0, 0)]^2 \\
& + \frac{m_\star^\ell}{(N - (n+1)\ell - i)(n\ell + i)} \sum_{x=2}^{\kappa-2} r(x, x+1) \frac{1}{\ell} \\
& \times [F(N - (n+1)\ell - i, x, \ell, 0) - F(N - (n+1)\ell - i, x, \ell, \ell)]^2 \\
& + \frac{1}{d_N L^2} \frac{r(\kappa - 1, \kappa)}{N - (n+1)\ell - i} \left(\sum_{k=1}^{\ell} \frac{1}{m_\star^k} \right)^{-1} \\
& \times [F(N - (n+1)\ell - i, \kappa - 1, 0, 0) - F(N - (n+1)\ell - i, \kappa - 1, \ell, 0)]^2 \bigg\} \\
\geq \frac{d_N^4(1+o(1))}{Z_N} \bigg\{ & \frac{d_N L^2 (n\ell + i)}{r(1,2)} \sum_{k=1}^{\ell} \frac{1}{m_\star^{\ell-k}} + \frac{\ell(N - (n+1)\ell - i)(n\ell + i)}{m_\star^\ell} \sum_{x=2}^{\kappa-2} \frac{1}{r(x, x+1)} \\
& \qquad + \frac{d_N L^2 (N - (n+1)\ell - i)}{r(\kappa - 1, \kappa)} \sum_{k=1}^{\ell} \frac{1}{m_\star^k} \bigg\}^{-1} \\
& \times [F(N - (n+1)\ell - i, \kappa - 1, 0, 0) - F(N - n\ell - i, 2, 0, 0)]^2 \\
\geq \frac{d_N^4(1+o(1))}{Z_N} & \frac{m_\star^\ell}{\ell(N - (n+1)\ell - i)(n\ell + i)} \left(\sum_{x=2}^{\kappa-2} \frac{1}{r(x, x+1)} \right)^{-1} \\
& \times [F(N - (n+1)\ell - i, 2, 0, 0) - F(N - n\ell - i, 2, 0, 0)]^2 , \qquad (6.8)
\end{aligned}$$

where the last equality holds since the first and third term are at most of order $d_N N$, whereas the second term is of order at least N. Using this bound on $A_{2,n}(\ell, i)$, we can bound $A_2(\ell, i)$ by applying Lemma 2.7 once more:

$$
\begin{aligned}
A_2(\ell,i) \geq & \sum_{n=0}^{\lfloor N/\ell\rfloor-2} \frac{d_N^4(1+o(1))}{Z_N} \frac{m_\star^\ell}{\ell(N-(n+1)\ell-i)(n\ell+i)} \left(\sum_{x=2}^{\kappa-2} \frac{1}{r(x,x+1)}\right)^{-1} \\
& \times \left[F(N-(n+1)\ell-i,\kappa-1,0,0)-F(N-n\ell-i,2,0,0)\right]^2 \\
\geq & \frac{d_N^4(1+o(1))}{Z_N} \frac{m_\star^\ell}{\ell} \left(\sum_{x=2}^{\kappa-2} \frac{1}{r(x,x+1)}\right)^{-1} \left(\sum_{n=0}^{\lfloor N/\ell\rfloor-2} (N-(n+1)\ell-i)(n\ell+i)\right)^{-1} \\
& \times \left[F(N-(\lfloor N/\ell\rfloor-1)\ell-i,2,0,0)-F(N-i,2,0,0)\right]^2 . \qquad (6.9)
\end{aligned}
$$

Using that

$$
\begin{aligned}
\sum_{n=0}^{\lfloor N/\ell\rfloor-2} (N-(n+1)\ell-i)(n\ell+i) &= N^3 \frac{1}{N} \sum_{n=0}^{\lfloor N/\ell\rfloor-2} \left(1-\frac{n+1}{N}\ell-\frac{i}{N}\right)\left(\frac{n}{N}\ell+\frac{i}{N}\right) \\
&= N^3(1+o(1)) \int_0^{1/\ell} (1-x\ell)x\ell \,\mathrm{d}x = \frac{N^3}{6\ell}(1+o(1)), \qquad (6.10)
\end{aligned}
$$

we get the bound

$$
\begin{aligned}
A_2(\ell,i) \geq & \, 6\frac{d_N^4(1+o(1))}{N^3 Z_N} m_\star^\ell \left(\sum_{x=2}^{\kappa-2} \frac{1}{r(x,x+1)}\right)^{-1} \\
& \times \left[F(N-(\lfloor N/\ell\rfloor-1)\ell-i,2,0,0)-F(N-i,2,0,0)\right]^2 . \qquad (6.11)
\end{aligned}
$$

By similar arguments, one can show that there exists a constant $C_1>0$, such that

$$
\begin{aligned}
A_1(\ell,i) = & \frac{1}{L^2} \sum_{n=0}^{i-1} \Big\{ \mu_N(N-n,2,0,0)(N-n)d_N r(1,2) \left[F(N-(n+1),2,1,1)-F(N-n,2,0,0)\right]^2 \\
& + \sum_{x=2}^{\kappa-2} \mu_N(N-(n+1),x,1,1) d_N r(x,x+1) \left[F(N-(n+1),x,1,0)-F(N-(n+1),x,1,1)\right]^2 \\
& + \mu_N(N-(n+1),\kappa-1,1,1)(d_N+n) \left[F(N-(n+1),\kappa-1,0,0)-F(N-(n+1),\kappa-1,1,1)\right]^2 \Big\} \\
\geq & \, C_1 \frac{d_N^4(1+o(1))}{Z_N N} \left[F(N-i,2,0,0)-F(N,2,0,0)\right]^2 . \qquad (6.12)
\end{aligned}
$$

One can also show that there exists a constant $C_3>0$, such that

$$
A_3(\ell,i) \geq C_3 \frac{d_N^4(1+o(1))}{Z_N N} \left[F(0,2,0,0)-F(N-(\lfloor N/\ell\rfloor-1)\ell-i,2,0,0)\right]^2 . \qquad (6.13)
$$

Hence, using Lemma 2.7 once more,

$$\begin{aligned}
D_N(F) \geq \sum_{\ell=1}^{L}\sum_{i=0}^{\ell-1}\Big\{ & C_1\frac{d_N^4(1+o(1))}{Z_N N}\left[F(N-i,2,0,0)-F(N,2,0,0)\right]^2 \\
& + 6\frac{d_N^4(1+o(1))}{N^3 Z_N} m_\star^\ell \left(\sum_{x=2}^{\kappa-2}\frac{1}{r(x,x+1)}\right)^{-1} \\
& \times \left[F(N-(\lfloor N/\ell\rfloor-1)\ell-i,2,0,0)-F(N-i,2,0,0)\right]^2 \\
& + C_3\frac{d_N^4(1+o(1))}{Z_N N}\left[F(0,2,0,0)-F(N-(\lfloor N/\ell\rfloor-1)\ell-i,2,0,0)\right]^2\Big\} \\
\geq \sum_{\ell=1}^{L}\sum_{i=0}^{\ell-1} & \left\{\frac{Z_N N(1+o(1))}{C_1 d_N^4}+\frac{N^3 Z_N(1+o(1))}{6 d_N^4}\frac{1}{m_\star^\ell}\sum_{x=2}^{\kappa-2}\frac{1}{r(x,x+1)}+\frac{Z_N N(1+o(1))}{C_3 d_N^4}\right\}^{-1} \\
& \times \left[F(0,2,0,0)-F(N,2,0,0)\right]^2 .
\end{aligned} \tag{6.14}$$

Note that the summands do not depend on i, and that $F(0,2,0,0) = 0$ and $F(N,2,0,0) = 1$. Therefore,

$$D_N(F) \geq 6\frac{d_N^4(1+o(1))}{N^3 Z_N}\left(\sum_{x=2}^{\kappa-2}\frac{1}{r(x,x+1)}\right)^{-1}\sum_{\ell=1}^{L}\ell m_\star^\ell. \tag{6.15}$$

Hence, it follows from Proposition 3.2 with $\kappa_\star = 2$ that,

$$\liminf_{N\to\infty}\frac{N^2}{d_N^3} \geq 3\left(\sum_{x=2}^{\kappa-2}\frac{1}{r(x,x+1)}\right)^{-1}\sum_{\ell=1}^{L}\ell m_\star^\ell, \tag{6.16}$$

and the proposition follows by taking the limit $L\to\infty$. □

Remark 6.2. *For systems where the $m_\star$ are not all equal, we obtain the correct order of magnitude, but the constant obtained is not the same as the one obtained in the upper bound in the next section, which we believe to be the correct one.*

On general graphs, this lower bound on the capacity between sites in $S_\star$ that are at graph distance at least three is also valid, since the Dirichlet form can always be restricted to only allow for jumps on one specific path, and then restricting the jumps further as in the proof. This proves that also in general systems longer time-scales cannot be present.

6.2 Upper bound on capacities

We have the following upper bound for general systems as in (2.21), with $\kappa \geq 4$, which coincides with the lower bound in the case that all the $m_\star$ are equal:

Proposition 6.3. *Let the underlying random walk be as in (2.21), with $\kappa \geq 4$. Furthermore, suppose that d_N decays subexponentially and $d_N \log N \to 0$ as $N\to\infty$. Then,*

$$\limsup_{N\to\infty}\frac{N^2}{d_N^3}\mathrm{Cap}_N\left(\mathcal{E}_N(1),\mathcal{E}_N(\kappa)\right) \leq 3\left(\sum_{x=2}^{\kappa-2}\frac{(1-m_\star(x))(1-m_\star(x+1))}{m_\star(x) r(x,x+1)}\right)^{-1}. \tag{6.17}$$

In particular, for $m_\star := m_\star(2) = \ldots = m_\star(\kappa-1)$,

$$\limsup_{N\to\infty} \frac{N^2}{d_N^3} \mathrm{Cap}_N\left(\mathcal{E}_N(1), \mathcal{E}_N(\kappa)\right) \le 3 \frac{m_\star}{(1-m_\star)^2} \left(\sum_{x=2}^{\kappa-2} \frac{1}{r(x,x+1)} \right)^{-1}. \tag{6.18}$$

Proof. From the lower bound, we can guess that a good test function should be of the form

$$F(\eta) = 6 \sum_{\ell=2}^{\kappa-2} c_\ell \int_0^{\phi_{2\varepsilon}\left(\frac{\eta_1}{N} + \left(\left(\frac{1}{N}\sum_{p=2}^{\ell} \eta_p\right) \wedge \varepsilon\right)\right)} x(1-x)\,\mathrm{d}x, \tag{6.19}$$

where we need to choose the constants such that

$$\sum_{\ell=2}^{\kappa-2} c_\ell = 1, \tag{6.20}$$

so that

$$F(\eta_1 = N) = 6 \sum_{\ell=2}^{\kappa-2} c_\ell \int_0^1 x(1-x)\,\mathrm{d}x = 1. \tag{6.21}$$

We obviously also have that $F(\eta_\kappa = N) = 0$, so that $F \in \mathcal{F}_N\left(\mathcal{E}_N(1), \mathcal{E}_N(\kappa)\right)$. We optimize over the constants c_ℓ at the end.

Because of the choice of $\phi_{2\varepsilon}$, we have that $F(\eta^{p,p+1}) - F(\eta) = 0$ for all $p = 1, \ldots, \kappa-1$ if $j < \varepsilon N$ or $j > (1-\varepsilon)N$. Denote by ℓ the total number of particles on sites $2, \ldots, \kappa-1$. Then, we have that, for $\ell < \varepsilon N$,

$$F(\eta^{1,2}) - F(\eta) = 0. \tag{6.22}$$

We also have, for all values of ℓ, that

$$F(\eta^{\kappa-1,\kappa}) - F(\eta) = 0. \tag{6.23}$$

Hence, also using reversibility in the first equality, we can rewrite the Dirichlet form of F as,

$$\begin{aligned} D_N(F) &= \sum_{\eta\in E_N} \mu_N(\eta) \sum_{q=1}^{\kappa-1} \eta_q(\eta_{q+1} + d_N) r(q,q+1) \left[F(\eta^{q,q+1}) - F(\eta)\right]^2 \\ &= \sum_{\ell=0}^{\varepsilon N-1} \sum_{j=\varepsilon N}^{(1-\varepsilon)N} \sum_{\eta_2+\ldots+\eta_{\kappa-1}=\ell} \mu_N(\eta) \sum_{q=2}^{\kappa-2} \eta_q(\eta_{q+1} + d_N) r(q,q+1) \left[F(\eta^{q,q+1}) - F(\eta)\right]^2 \\ &\quad + \sum_{\ell=\varepsilon N}^{N} \sum_{j=\varepsilon N}^{N-\ell} \sum_{\eta_2+\ldots+\eta_{\kappa-1}=\ell} \mu_N(\eta) \sum_{q=1}^{\kappa-2} \eta_q(\eta_{q+1} + d_N) r(q,q+1) \left[F(\eta^{q,q+1}) - F(\eta)\right]^2. \end{aligned} \tag{6.24}$$

For small ℓ, we split the sum

$$\sum_{\eta_2+\ldots+\eta_{\kappa-1}=\ell} = \sum_{p=2}^{\kappa-2} \sum_{\eta_p=1}^{\ell} \mathbb{1}\{\eta_{p+1} = \ell - \eta_p\} + \sum_{\substack{\eta_2+\ldots+\eta_{\kappa-1}=\ell \\ \eta_p+\eta_{p+1}<\ell\,\forall\, 2\le p\le\kappa-2}}. \tag{6.25}$$

The first sum consists of all configurations with ℓ particles on at most 2 adjacent sites in $\{2,\dots,\kappa-1\}$, and with the rest of the particles only on sites 1 and κ, while the second sum consists of all other configurations. This latter sum turns out to have a negligible contribution, as we show later. Let us first analyze the first sum:

$$\begin{aligned}
&\sum_{\ell=0}^{\varepsilon N-1}\sum_{j=\varepsilon N}^{(1-\varepsilon)N}\sum_{p=2}^{\kappa-2}\sum_{\eta_p=1}^{\ell}\mathbb{1}\{\eta_{p+1}=\ell-\eta_p\}\mu_N(\eta)\sum_{q=2}^{\kappa-2}\eta_q(\eta_{q+1}+d_N)r(q,q+1)\left[F(\eta^{q,q+1})-F(\eta)\right]^2\\
&\quad=\frac{1}{Z_N}\sum_{\ell=0}^{\varepsilon N-1}\sum_{j=\varepsilon N}^{(1-\varepsilon)N}w_N(j)w_N(N-j-\ell)\sum_{p=2}^{\kappa-2}\sum_{\eta_p=1}^{\ell}w_N(\eta_p)m_\star(p)^\ell w_N(\ell-\eta_p)m_\star(p+1)^{\ell-\eta_p}\\
&\qquad\qquad\sum_{q=p}^{(p+1)\wedge(\kappa-2)}\eta_q(\eta_{q+1}+d_N)r(q,q+1)\left[F(\eta^{q,q+1})-F(\eta)\right]^2,
\end{aligned}\tag{6.26}$$

since all other q give a 0 contribution, because then $\eta_q=0$.

Using Lemma 3.1, the above equals

$$\begin{aligned}
&\frac{d_N^4}{Z_N}(1+o(1))\sum_{\ell=0}^{\varepsilon N-1}\sum_{j=\varepsilon N}^{(1-\varepsilon)N}\frac{1}{j(N-j-\ell)}\sum_{p=2}^{\kappa-2}\sum_{\eta_p=1}^{\ell}m_\star(p)^{\eta_p}m_\star(p+1)^{\ell-\eta_p}\\
&\qquad\left(r(p,p+1)\left[F(\eta^{p,p+1})-F(\eta)\right]^2+\frac{d_N}{\eta_p}r(p+1,p+2)\left[F(\eta^{p+1,(p+2)\wedge(\kappa-2)})-F(\eta)\right]^2\right)\\
&=6^2\frac{d_N^4}{Z_N}(1+o(1))\sum_{\ell=0}^{\varepsilon N-1}\sum_{p=2}^{\kappa-2}\sum_{\eta_p=1}^{\ell}m_\star(p)^{\eta_p}m_\star(p+1)^{\ell-\eta_p}\sum_{j=\varepsilon N}^{(1-\varepsilon)N}\frac{1}{j(N-j-\ell)}\\
&\qquad\left(r(p,p+1)\left[c_p\int_{\phi_{2\varepsilon}\left(\frac{j+\eta_p-1}{N}\right)}^{\phi_{2\varepsilon}\left(\frac{j+\eta_p}{N}\right)}x(1-x)\,\mathrm{d}x\right]^2\right.\\
&\qquad\qquad\left.+\frac{d_N}{\eta_p}r(p+1,p+2)\left[c_{p+1}\int_{\phi_{2\varepsilon}\left(\frac{j+\eta_{p+1}-\mathbb{1}\{p<\kappa-2\}}{N}\right)}^{\phi_{2\varepsilon}\left(\frac{j+\eta_{p+1}}{N}\right)}x(1-x)\,\mathrm{d}x\right]^2\right).
\end{aligned}\tag{6.27}$$

Similarly to the upper bound in the second time-scale, it holds that

$$\begin{aligned}
&\sum_{j=\varepsilon N}^{(1-\varepsilon)N}\frac{1}{j(N-j-\ell)}\left[\int_{\phi_{2\varepsilon}\left(\frac{j+\eta_p-1}{N}\right)}^{\phi_{2\varepsilon}\left(\frac{j+\eta_p}{N}\right)}x(1-x)\,\mathrm{d}x\right]^2\\
&\qquad=\frac{1}{N^2}\sum_{j=\varepsilon N}^{(1-\varepsilon)N}\int_{\phi_{2\varepsilon}\left(\frac{j+\eta_p-1}{N}\right)}^{\phi_{2\varepsilon}\left(\frac{j+\eta_p}{N}\right)}x(1-x)\,\mathrm{d}x\int_{\phi_{2\varepsilon}\left(\frac{j+\eta_p-1}{N}\right)}^{\phi_{2\varepsilon}\left(\frac{j+\eta_p}{N}\right)}\frac{x}{j/N}\frac{1-x}{1-(j+\ell)/N}\,\mathrm{d}x\\
&\qquad\le\frac{(1+\sqrt{\varepsilon})^3}{N^3}\int_0^1x(1-x)\,\mathrm{d}x=\frac{(1+\sqrt{\varepsilon})^3}{6N^3}.
\end{aligned}\tag{6.28}$$

Hence, (6.27) is bounded from above by

$$6\frac{(1+\sqrt{\varepsilon})^3}{N^3}\frac{d_N^4}{Z_N}(1+o(1))\sum_{p=2}^{\kappa-2} c_p^2 r(p,p+1)\sum_{\ell=0}^{\varepsilon N-1}\sum_{\eta_p=1}^{\ell} m_\star(p)^{\eta_p} m_\star(p+1)^{\ell-\eta_p}, \tag{6.29}$$

because the contribution of the second part of the last line of (6.27) is clearly $o(1)$ times the contribution of the first part.

To bound the contribution of the last sum in (6.25), we set $M = \max_{v\notin S_\star} m_\star(v)$ and observe that, for all such configurations and all q,

$$m_\star^\eta w_N(\eta)\eta_q(\eta_{q+1}+d_N) \le M^\ell \frac{d_N^5}{j(N-j-\ell)}, \tag{6.30}$$

because either at least 5 sites are occupied, or 4 sites are occupied but $\eta_{q+1}=0$. Then one can show, as above, that this contribution is also negligible compared to (6.29).

To show that the sum over $\ell \ge \varepsilon N$ in (6.24) is negligible, we write

$$\sum_{q=1}^{\kappa-2} \eta_q(\eta_{q+1}+d_N) r(q,q+1) \le (\kappa-3)RN^2, \tag{6.31}$$

where we set $R = \max_{\ell=2}^{\kappa-2} r(\ell,\ell+1)$. Furthermore, $\left[F(\eta^{q,q+1})-F(\eta)\right]^2 \le 1$ and

$$\mu_N(\eta) \le \frac{M^{\varepsilon N}}{Z_N} w_N(\eta) = \frac{M^{\varepsilon N} N}{2d_N}(1+o(1))w_N(\eta). \tag{6.32}$$

Hence,

$$\begin{aligned}&\sum_{\ell=\varepsilon N}^{N}\sum_{j=\varepsilon N}^{N-\ell}\sum_{\eta_2+\ldots+\eta_{\kappa-1}=\ell} \mu_N(\eta)\sum_{q=1}^{\kappa-2}\eta_q(\eta_{q+1}+d_N)r(q,q+1)\left[F(\eta^{q,q+1})-F(\eta)\right]^2\\ &\qquad\le (\kappa-3)R\frac{M^{\varepsilon N}N^3}{2d_N}(1+o(1))\sum_{\ell=\varepsilon N}^{N}\sum_{j=\varepsilon N}^{N-\ell}\sum_{\eta_2+\ldots+\eta_{\kappa-1}=\ell} w_N(\eta).\end{aligned} \tag{6.33}$$

Now we can write

$$\sum_{\ell=\varepsilon N}^{N}\sum_{j=\varepsilon N}^{N-\ell}\sum_{\eta_2+\ldots+\eta_{\kappa-1}=\ell} w_N(\eta) \le \sum_{\ell=0}^{N}\sum_{j=0}^{N-\ell}\sum_{\eta_2+\ldots+\eta_{\kappa-1}=\ell} w_N(\eta) = \tilde{Z}_N, \tag{6.34}$$

where $\tilde{Z}_N$ is the partition function of a similar system where we set $m_\star(v)=1$ for all $v\in\{1,\ldots,\kappa\}$. Hence,

$$\tilde{Z}_N = \frac{\kappa d_N}{N}(1+o(1)). \tag{6.35}$$

and we get

$$\begin{aligned}&\frac{N^2}{d_N^3}\sum_{\ell=\varepsilon N}^{N}\sum_{j=0}^{N-\ell}\sum_{\eta_2+\ldots+\eta_{\kappa-1}=\ell} \mu_N(\eta)\sum_{q=1}^{\kappa-2}\eta_q(\eta_{q+1}+d_N)r(q,q+1)\left[F(\eta^{q,q+1})-F(\eta)\right]^2\\ &\qquad\le \kappa(\kappa-3)R\frac{M^{\varepsilon N}N^4}{2d_N^3}(1+o(1)),\end{aligned} \tag{6.36}$$

which converges to 0 because d_N decays subexponentially.

Thus, the only significant contribution to $D_N(F)$ can be bounded from above by (6.29), and altogether we obtain

$$\begin{aligned}\limsup_{N\to\infty} \frac{N^2}{d_N^3} D_N(F) &\le \limsup_{N\to\infty}\, 6(1+\sqrt{\varepsilon})^3 \frac{d_N}{NZ_N}(1+o(1)) \sum_{p=2}^{\kappa-2} c_p^2 r(p,p+1) \sum_{\ell=0}^{\varepsilon N-1} \sum_{\eta_p=1}^{\ell} m_\star(p)^{\eta_p} m_\star(p+1)^{\ell-\eta_p} \\ &= 3(1+\sqrt{\varepsilon})^3 \sum_{p=2}^{\kappa-2} c_p^2 r(p,p+1) m_\star(p) \sum_{\ell=0}^{\infty} \sum_{\eta_p=0}^{\ell-1} m_\star(p)^{\eta_p} m_\star(p+1)^{\ell-1-\eta_p} \\ &= 3(1+\sqrt{\varepsilon})^3 \sum_{p=2}^{\kappa-2} c_p^2 \frac{r(p,p+1) m_\star(p)}{(1-m_\star(p))(1-m_\star(p+1))}. \end{aligned} \tag{6.37}$$

We finally optimize over the constants c_p. Let us write $c_p = g(p) - g(p+1)$. By (6.20), we need that

$$\sum_{p=2}^{\kappa-2} c_p = \sum_{p=2}^{\kappa-2} (g(p) - g(p+1)) = g(2) - g(\kappa-1) = 1, \tag{6.38}$$

and hence, without loss of generality, we can optimize over functions g such that $g(2) = 1$ and $g(\kappa-1) = 0$. Then, taking the infimum over all such functions g, it follows from (6.37) that

$$\begin{aligned}\limsup_{N\to\infty} \frac{N^2}{d_N^3} D_N(F) &\le \inf_{g: g(2)=1, g(\kappa-2)=0} 3(1+\sqrt{\varepsilon})^3 \sum_{p=2}^{\kappa-2} [g(p)-g(p+1)]^2 \frac{r(p,p+1) m_\star(p)}{(1-m_\star(p))(1-m_\star(p+1))} \\ &= 3(1+\sqrt{\varepsilon})^3 \left(\sum_{p=2}^{\kappa-2} \frac{(1-m_\star(p))(1-m_\star(p+1))}{r(p,p+1) m_\star(p)} \right)^{-1}, \end{aligned} \tag{6.39}$$

because this is again the effective capacity of a linear chain.

The first statement of the proposition now follows by taking $\varepsilon \to 0$. The second statement easily follows from the first. □

6.3 Proof of Theorem 2.6

The proof again runs similarly to that of Theorem 2.3.

Proof of Theorem 2.6(i). As a consequence of Propositions 6.1 and 6.3, if d_N decays subexponentially and $d_N \log N \to 0$ as $N \to \infty$ and $m_\star(2) = \ldots = m_\star(\kappa-1) < 1$,

$$\lim_{N\to\infty} \frac{N^2}{d_N^3} \mathrm{Cap}_N\left(\mathcal{E}_N(1), \mathcal{E}_N(\kappa)\right) = 3 \frac{m_\star}{(1-m_\star)^2} \left(\sum_{x=2}^{\kappa-2} \frac{1}{r(x,x+1)} \right)^{-1}. \tag{6.40}$$

In view of (2.33), recalling that $\mathrm{Cap}_N(A,B) = \mathrm{Cap}_N(B,A)$, and applying (4.45), this provides formula (2.25). □

Proof of Theorem 2.6(ii). As in the proof of Theorem 2.3(ii), the convergence follows from Theorem 2.7 of [3] once condition (2.36) of Section 2.5 is verified for the sequence $\theta_N = N^2/d_N^3$, $N \geq 1$. By Lemma 6.8 of [3] (see (2.37) in Section 2.5) and using Proposition 2.1 and (6.40), we get

$$\lim_{N\to\infty} \frac{N^2}{d_N^3} r_N^{\mathcal{E}_\star}\left(\mathcal{E}_N^1, \mathcal{E}_N^\kappa\right) = \lim_{N\to\infty} \frac{N^2}{d_N^3} r_N^{\mathcal{E}_\star}\left(\mathcal{E}_N^\kappa, \mathcal{E}_N^1\right) = 3\frac{m_\star}{(1-m_\star)^2}\left(\sum_{x=2}^{\kappa-2} \frac{1}{r(x,x+1)}\right)^{-1}, \tag{6.41}$$

proving (2.36). Finally, (2.27) is proved similarly to (2.20). □

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