Ising critical exponents on random trees and graphs

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Abstract

We study the critical behavior of the ferromagnetic Ising model on random trees as well as so-called locally tree-like random graphs. We pay special attention to trees and graphs with a power-law offspring or degree distribution whose tail behavior is characterized by its power-law exponent $\tau > 2$. We show that the critical inverse temperature of the Ising model equals the hyperbolic arctangent of the reciprocal of the mean offspring or mean forward degree distribution. In particular, the critical inverse temperature equals zero when $\tau \in (2, 3]$ where this mean equals infinity.

We further study the critical exponents $\boldsymbol{\delta}, \boldsymbol{\beta}$ and $\boldsymbol{\gamma}$, describing how the (root) magnetization behaves close to criticality. We rigorously identify these critical exponents and show that they take the values as predicted by Dorogovstev, et al. [16] and Leone et al. [26]. These values depend on the power-law exponent τ, taking the same values as the mean-field Curie-Weiss model [2] for $\tau > 5$, but different values for $\tau \in (3, 5)$.

1 Introduction

In the past decades complex networks and their behavior have attracted much attention. In the real world many of such networks can be found, for instance as social, information, technological and biological networks. An interesting property that many of them share is that they are *scale free* [31]. This means that their degree sequences obey a *power law*, i.e., the fraction of nodes that have k neighbors is proportional to $k^{-\tau}$ for some $\tau > 1$. We therefore use power-law random graphs as a simple model for real-world networks. Examples of how to generate such random graphs can, e.g., be found in [8].

Not only the structure of these networks is interesting, also the behavior of processes living on these networks is a fascinating subject. Processes one can think of are opinion formation, the spread of information and the spread of viruses. An extensive overview of complex networks and processes on them is given by Newman in [31]. It is especially interesting if these processes undergo a so-called *phase transition*, i.e., a minor change in the circumstances suddenly results in

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completely different behavior (see [10] for a mathematical survey on Gibbs measures and phase transitions on sparse random graphs).

Physicists have studied the behavior near phase transitions, the *critical behavior*, on complex networks for many different models, see [17] for an overview. Many of these results have not been rigorously proved. One of the few models for which rigorous results have been obtained is the contact process [9], where the predictions of physicists, in fact, turned out not to be correct. A mathematical treatment of other models is therefore necessary.

We focus on the *Ising model*, a classical model for the study of phase transitions [32, 33, 34]. In this model a spin value that can be either $+1$ or -1 is assigned to every vertex. These spins influence each other with *ferromagnetic* interactions, i.e., neighboring spins prefer to be aligned. The strength of these interactions depends on the temperature. The first rigorous study of the Ising model on a random graph was performed by De Sanctis and Guerra in [35], where the high and zero temperature regime of the Ising model on the Erdős-Rényi random graph were analyzed. Later, in [11], Dembo and Montanari analyzed the Ising model for any temperature on random graphs with a finite-variance degree distribution that converge locally to a Galton-Watson tree. In [15], we generalized these results to the case where the degree distribution has strongly finite mean, but possibly infinite variance, i.e., the degree distribution obeys a power-law with exponent $\tau > 2$. In [13], Dembo, Montanari and Sun generalize this further to more general locally tree-like graphs. An analysis of the critical behavior, however, was still lacking.

In this article, we rigorously study the critical behavior of the Ising model on power-law random graphs by computing the critical temperature of the model and the critical exponents describing the scaling of two thermodynamic quantities (the magnetization and the susceptibility) around the critical point. More precisely, we compute the exponent $\boldsymbol{\delta}$ that describes the behavior of the magnetization at the critical temperature as the external field vanishes, the exponent $\boldsymbol{\beta}$ that describes the behavior of the spontaneous magnetization as the temperature increases to the critical temperature and the exponent $\boldsymbol{\gamma}$ that describes the behavior of the susceptibility as the temperature decreases to the critical temperature. We also provide an heuristic lower bound for the exponent $\boldsymbol{\gamma'}$ describing the divergence of susceptibility as the critical temperature is approached from below. Predictions for the values of these exponents were given by Dorogovtsev et al. in [16] and independently by Leone et al. in [26] and we prove that these values are indeed correct. These exponents depend on the power-law exponent τ. We prove that the critical exponents $\boldsymbol{\delta}, \boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ take the same values as the mean-field Curie-Weiss model for $\tau > 5$, and hence also for the Erdős-Rényi random graph, but are different for $\tau \in (3, 5)$. In [16, 26] also the case $\tau \in (2, 3)$ is studied for which the critical temperature is infinite. Hence, the critical behavior should be interpreted as the temperature going to infinity, which is a different problem from approaching a finite critical temperature and is therefore beyond the scope of this article.

Our proofs always start by relating the magnetization of the Ising model on the random graph and various of its derivatives to the root magnetization of a rooted random tree, the so-called unimodular Galton-Watson tree, see [5] for a detailed discussion of local convergence of random graphs to unimodular trees. After this, we identify the critical exponents related to the root magnetization on the rooted random tree. As a result, all our results also apply to this setting, where only in the case of the regular tree, the mean-field critical exponents have been identified [2], and which we extend to general offspring distributions.

2 Model definitions and results

2.1 Ising model on finite graphs

We start by defining Ising models on finite graphs. Consider a random graph sequence $(G_n)_{n\geq 1}$. Here $G_n = (V_n, E_n)$, with vertex set $V_n = [n] \equiv \{1, \ldots, n\}$ and with a random edge set E_n. To each vertex $i \in [n]$ an Ising spin $\sigma_i = \pm 1$ is assigned. A configuration of spins is denoted by $\sigma = (\sigma_i)_{i\in[n]}$. The *Ising model on* G_n is then defined by the Boltzmann-Gibbs measure

$$\mu_n(\sigma) = \frac{1}{Z_n(\beta, \underline{B})} \exp\left\{ \beta \sum_{(i,j)\in E_n} \sigma_i\sigma_j + \sum_{i\in[n]} B_i\sigma_i \right\}. \tag{2.1}$$

Here, $\beta \geq 0$ is the inverse temperature and $\underline{B}$ the vector of external magnetic fields $\underline{B} = (B_i)_{i\in[n]} \in \mathbb{R}^n$. For a uniform external field we write B instead of $\underline{B}$, i.e., $B_i = B$ for all $i \in [n]$. The partition function $Z_n(\beta, \underline{B})$ is the normalization constant in (2.1), i.e.,

$$Z_n(\beta, \underline{B}) = \sum_{\sigma\in\{-1,+1\}^n} \exp\left\{ \beta \sum_{(i,j)\in E_n} \sigma_i\sigma_j + \sum_{i\in[n]} B_i\sigma_i \right\}. \tag{2.2}$$

Note that the inverse temperature β does not multiply the external field. This turns out to be technically convenient and does not change the results, because we are only looking at systems at equilibrium, and hence this would just be a reparametrization.

We let $\langle \cdot \rangle_{\mu_n}$ denote the expectation with respect to the Ising measure μ_n, i.e., for every bounded function $f : \{-1, +1\}^n \to \mathbb{R}$, we write

$$\langle f(\sigma) \rangle_{\mu_n} = \sum_{\sigma\in\{-1,+1\}^n} f(\sigma)\mu_n(\sigma). \tag{2.3}$$

2.2 Thermodynamics

We study the critical behavior of this Ising model by analyzing the following two thermodynamic quantities:

Definition 2.1 (Thermodynamic quantities)**.** For a graph sequence $(G_n)_{n\geq 1}$,

(a) let $M_n(\beta, B) = \frac{1}{n}\sum_{i\in[n]}\langle\sigma_i\rangle_{\mu_n}$ be the magnetization per vertex. Then, the thermodynamic limit of the *magnetization* per vertex equals

$$M(\beta, B) \equiv \lim_{n\to\infty} M_n(\beta, B). \tag{2.4}$$

(b) let $\chi_n(\beta, B) = \frac{1}{n}\sum_{i,j\in[n]}\left(\langle\sigma_i\sigma_j\rangle_{\mu_n} - \langle\sigma_i\rangle_{\mu_n}\langle\sigma_j\rangle_{\mu_n}\right)$ denote the susceptibility. Then, the thermodynamic limit of the *susceptibility* equals

$$\chi(\beta, B) \equiv \lim_{n\to\infty} \chi_n(\beta, B). \tag{2.5}$$

The existence of the above limits for $n \to \infty$ has been proved in [15, Theorem 1.5], using the existence of the pressure per particle proved in [11] and [15, Theorem 1.4] and using monotonicity properties. We now define the critical temperature. We write $f(0^+)$ for $\lim_{x \searrow 0} f(x)$.

Definition 2.2 (Critical inverse temperature)**.** The critical inverse temperature equals

$$\beta_c \equiv \inf\{\beta : M(\beta, 0^+) > 0\}. \tag{2.6}$$

By the GKS inequalities [24], $M(\beta, B)$ is non-negative and non-decreasing in B for $B \geq 0$ so that the limit $M(\beta, 0^+)$ indeed exists. Note that β_c can only exist in the thermodynamic limit, but not for the magnetization of a finite graph, since always $M_n(\beta, 0^+) = 0$. The critical behavior can now be expressed in terms of the following critical exponents. We write $f(x) \asymp g(x)$ if the ratio $f(x)/g(x)$ is bounded away from 0 and infinity for the specified limit.

Definition 2.3 (Critical exponents)**.** The critical exponents $\boldsymbol{\beta}, \boldsymbol{\delta}, \boldsymbol{\gamma}, \boldsymbol{\gamma'}$ are defined by:

$$M(\beta, 0^+) \asymp (\beta - \beta_c)^{\boldsymbol{\beta}}, \qquad \text{for } \beta \searrow \beta_c; \tag{2.7}$$

$$M(\beta_c, B) \asymp B^{1/\boldsymbol{\delta}}, \qquad \text{for } B \searrow 0; \tag{2.8}$$

$$\chi(\beta, 0^+) \asymp (\beta_c - \beta)^{-\boldsymbol{\gamma}}, \qquad \text{for } \beta \nearrow \beta_c; \tag{2.9}$$

$$\chi(\beta, 0^+) \asymp (\beta - \beta_c)^{-\boldsymbol{\gamma'}}, \qquad \text{for } \beta \searrow \beta_c. \tag{2.10}$$

Remark. *The definitions above are meaningful since for $\beta > \beta_c$ one has $M(\beta, 0^+) \neq M(\beta, 0) = 0$, i.e., the magnetization of the low-temperature phase is discontinuous in $B = 0$.*

We emphasize that there is a difference between the symbol β for the inverse temperature and the bold symbol $\boldsymbol{\beta}$ for the critical exponent in (2.7)*. Both notations are standard in the literature, so we decided to follow both of them and distinguish them by the font style.*

Also note that these are stronger definitions than usual. E.g., normally the critical exponent $\boldsymbol{\beta}$ is defined as that value such that

$$M(\beta, 0^+) = (\beta - \beta_c)^{\boldsymbol{\beta} + o(1)}, \tag{2.11}$$

where $o(1)$ is a function tending to zero for $\beta \searrow \beta_c$.

2.3 Locally tree-like random graphs

We study the critical behavior of the Ising model on graph sequences $(G_n)_{n \geq 1}$ that are assumed to be *locally like a homogeneous random tree* [10, 3], to have a *power-law degree distribution* and to be *uniformly sparse*. We give the formal definitions of these assumptions below, but we first introduce some notation.

Let the random variable D have distribution $P = (p_k)_{k \geq 0}$, i.e., $\mathbb{P}[D = k] = p_k$, for $k = 0, 1, 2, \ldots$. We define its *forward degree distribution* by

$$\rho_k = \frac{(k+1) p_{k+1}}{\mathbb{E}[D]}, \tag{2.12}$$

where we assume that $\mathbb{E}[D] < \infty$. Let K be a random variable with $\mathbb{P}[K = k] = \rho_k$ and write $\nu = \mathbb{E}[K]$. The random rooted tree $\mathcal{T}(D, K, \ell)$ is a branching process with ℓ generations, where the root offspring is distributed as D and the vertices in each next generation have offsprings that are independent of the root offspring and are *independent and identically distributed* (i.i.d.) copies of the random variable K. We write $\mathcal{T}(K,\ell)$ when the offspring at the root has the same distribution as K.

We write that an event $\mathcal{A}$ holds *almost surely* (a.s.) if $\mathbb{P}[\mathcal{A}] = 1$. If $\nu \leq 1$ the branching processes $\mathcal{T}(D,K,\ell)$ and $\mathcal{T}(K,\ell)$ die out a.s. and the random graph sequence $(G_n)_{n\geq 1}$ does not have a giant component a.s. [23]. Therefore, there are no phase transitions when $\nu \leq 1$ and thus we assume that $\nu > 1$ throughout the rest of the paper.

The ball of radius r around vertex i, $B_i(r)$, is defined as the graph induced by the vertices at graph distance at most r from vertex i. For two rooted trees $\mathcal{T}_1$ and $\mathcal{T}_2$, we write that $\mathcal{T}_1 \simeq \mathcal{T}_2$, when there exists a bijective map from the vertices of $\mathcal{T}_1$ to those of $\mathcal{T}_2$ that preserves the adjacency relations.

Definition 2.4 (Local convergence to homogeneous random trees)**.** *Let $\mathbb{P}_n$ denote the law induced on the ball $B_i(t)$ in G_n centered at a uniformly chosen vertex $i \in [n]$. We say that the graph sequence $(G_n)_{n\geq 1}$ is* locally tree-like *with asymptotic degree distribution P when, for any rooted tree $\mathcal{T}$ with t generations*

$$\lim_{n\to\infty} \mathbb{P}_n[B_i(t) \simeq \mathcal{T}] = \mathbb{P}[\mathcal{T}(D,K,t) \simeq \mathcal{T}]. \tag{2.13}$$

Note that this implies in particular that the degree of a uniformly chosen vertex of the graph has an asymptotic degree distributed as D.

Definition 2.5 (Uniform sparsity)**.** *We say that the graph sequence $(G_n)_{n\geq 1}$ is* uniformly sparse *when, a.s.,*

$$\lim_{\ell\to\infty} \limsup_{n\to\infty} \frac{1}{n} \sum_{i\in[n]} D_i \mathbb{1}_{\{D_i\geq \ell\}} = 0, \tag{2.14}$$

where D_i is the degree of vertex i and $\mathbb{1}_{\mathcal{A}}$ denotes the indicator of the event $\mathcal{A}$.

Note that uniform sparsity follows if $\frac{1}{n}\sum_{i\in[n]} D_i \to \mathbb{E}[D]$ a.s., by the weak convergence of the degree of a uniform vertex.

We pay special attention to cases where the degree distribution satisfies a power law, as defined below. For power-law degree distributions, not all moments of the degrees are finite, which has severe consequences for the critical behavior of the Ising model.

Definition 2.6 (Power laws)**.** *We say that the distribution $P = (p_k)_{k\geq 0}$ obeys a* power law with exponent τ *when there exist constants $C_p > c_p > 0$ such that, for all $k = 1, 2, \ldots$,*

$$c_p k^{-(\tau-1)} \leq \sum_{\ell\geq k} p_\ell \leq C_p k^{-(\tau-1)}. \tag{2.15}$$

2.4 The random Bethe tree

We next extend our definitions to the random tree $\mathcal{T}(D,K,\infty)$, which is an infinite random tree. One has to be very careful in defining a Gibbs measure on this tree, since trees suffer from the fact that the boundaries of intrinsic (i.e., graph distance) balls in them have a size that is comparable to their volume. We can adapt the construction of the Ising model on the regular tree in [2] to this setting, as we now explain. For $\beta \geq 0, B \in \mathbb{R}$, let $\mu_{t,\beta,B}^{+/f}$ be the Ising model on $\mathcal{T}(D, K, t)$ with $+$ respectively free boundary conditions. For a function f that only depends on $\mathcal{T}(D, K, m)$ with $m \leq t$, we let

$$\langle f \rangle_{\mu_{\beta,B}^{+/f}} = \lim_{t\to\infty} \langle f \rangle_{\mu_{t,\beta,B}^{+/f}}. \tag{2.16}$$

These limits indeed exist and are equal for $B > 0$ [11, 15]. This defines a unique infinite volume Gibbs measure $\mu_{\beta,B}^{+/f}$ on the random Bethe tree. Denoting by ϕ the root of the unimodular Galton-Watson random tree, the quantity $M(\beta, B)$ is defined as the expected root magnetization for the infinite volume Gibbs measure on the random Bethe tree:

$$M(\beta, B) = \mathbb{E}\left[\langle \sigma_\phi \rangle_{\mu_{\beta,B}^{+/f}}\right]. \tag{2.17}$$

Analogously, the susceptibility of the random Bethe tree is defined as the sum of the expected edge correlations, i.e.,

$$\chi(\beta, B) = \mathbb{E}\left[\sum_{j\in\mathcal{T}(D,K,\infty)} \left(\langle \sigma_\phi \sigma_j \rangle_{\mu_{\beta,B}^{+/f}} - \langle \sigma_\phi \rangle_{\mu_{\beta,B}^{+/f}} \langle \sigma_j \rangle_{\mu_{\beta,B}^{+/f}}\right)\right]. \tag{2.18}$$

Our results also apply to this setting under the assumption that the degree of the root obeys a power law in (2.15) or that $\mathbb{E}[K^3] < \infty$. The critical value β_c for the root magnetization is again defined by (2.6).

2.5 Main results

We now present our main results which describe the critical behavior of the Ising model on power-law random graphs and random trees with power-law offspring distribution. We first give an expression for the critical temperature:

Theorem 2.7 (Critical temperature)**.** *Assume that the random graph sequence $(G_n)_{n\geq 1}$ is locally tree-like with asymptotic degree distribution P and is uniformly sparse. Then, a.s., the critical temperature β_c of $(G_n)_{n\geq 1}$ and of the random Bethe tree $\mathcal{T}(D, K, \infty)$ equals*

$$\beta_c = \operatorname{atanh}(1/\nu). \tag{2.19}$$

Note that if $\nu \searrow 1$ then $\beta_c \to \infty$ which is to be expected as there is no phase transition for $\nu \leq 1$ at any positive temperature. The other extreme is when $\nu = \infty$, which is the case, e.g., if the degree distribution obeys a power law with exponent $\tau \in (2, 3]$. In that case $\beta_c = 0$ and hence the spontaneous magnetization is positive for any finite temperature.

Near the critical temperature the behavior of the Ising model can be described by critical exponents. The values of these critical exponents for different values of τ are stated in the following theorem:

Theorem 2.8 (Critical exponents)**.** *Assume one of the following condition is satisfied:*

(a) a random graph sequence $(G_n)_{n\geq 1}$ is given. The sequence is locally tree-like with asymptotic degree distribution P that obeys $\mathbb{E}[K^3] < \infty$ or a power law with exponent $\tau \in (3, 5]$, and it is uniformly sparse. Let $M(\beta, B)$ and $\chi(\beta, B)$ be as defined in (2.4) *and* (2.5)*;*

(b) a random Bethe tree that obeys $\mathbb{E}[K^3] < \infty$ or a power law with exponent $\tau \in (3, 5]$ is given. Let $M(\beta, B)$ and $\chi(\beta, B)$ be as defined in (2.17) and (2.18) .

Then, the critical exponents $\boldsymbol{\beta}$, $\boldsymbol{\delta}$ and $\boldsymbol{\gamma}$ defined in Definition 2.3 for the case (a) and the analogous exponents for the case (b) exist and satisfy

	$\tau \in (3, 5)$	$\mathbb{E}[K^3] < \infty$
$\boldsymbol{\beta}$	$1/(\tau - 3)$	$1/2$
$\boldsymbol{\delta}$	$\tau - 2$	3
$\boldsymbol{\gamma}$	1	1

The exponent $\boldsymbol{\gamma}'$ defined in Definition 2.3 for the case (a), as well as the analogous exponent for the case (b), satisfies $\boldsymbol{\gamma}' \geq 1$.

For the boundary case $\tau = 5$ there are logarithmic corrections for $\boldsymbol{\beta} = 1/2$ and $\boldsymbol{\delta} = 3$, but not for $\boldsymbol{\gamma} = 1$ and for the lower bound $\boldsymbol{\gamma}' \geq 1$. Indeed, (2.9) holds with $\boldsymbol{\gamma} = 1$ and the lower bound in (2.10) holds with $\boldsymbol{\gamma}' = 1$, while

$$M(\beta, 0^+) \asymp \Big(\frac{\beta - \beta_c}{\log 1/(\beta - \beta_c)}\Big)^{1/2} \quad \text{for } \beta \searrow \beta_c, \qquad M(\beta_c, B) \asymp \Big(\frac{B}{\log(1/B)}\Big)^{1/3} \quad \text{for } B \searrow 0. \tag{2.20}$$

Our results show that $\chi(\beta, 0^+) \geq c(\beta - \beta_c)^{-1}$ for some constant $c > 0$, which, if the critical exponent $\boldsymbol{\gamma}'$ exists, then it must satisfy $\boldsymbol{\gamma}' \geq 1$. See Proposition 6.2. Unfortunately, we cannot prove that the critical exponent $\boldsymbol{\gamma}'$ exists, see the discussion in the next section for more details on this issue.

From the previous theorem we can also derive the joint scaling of the magnetization as $(\beta, B) \searrow (\beta_c, 0)$:

Corollary 2.9 (Joint scaling in B and $(\beta - \beta_c)$)**.** *Under the conditions of Theorem 2.8 with $\tau \neq 5$,*

$$M(\beta, B) = \Theta\big((\beta - \beta_c)^{\boldsymbol{\beta}} + B^{1/\boldsymbol{\delta}}\big), \tag{2.21}$$

where $f(\beta, B) = \Theta(g(\beta, B))$ means that there exist constants $c_1, C_1 > 0$ such that $c_1 g(\beta, B) \leq f(\beta, B) \leq C_1 g(\beta, B)$ for all $B \in (0, \varepsilon)$ and $\beta \in (\beta_c, \beta_c + \varepsilon)$ with ε small enough.

For $\tau = 5$,

$$M(\beta, B) = \Theta\Big(\Big(\frac{\beta - \beta_c}{\log 1/(\beta - \beta_c)}\Big)^{1/2} + \Big(\frac{B}{\log(1/B)}\Big)^{1/3}\Big). \tag{2.22}$$

2.6 Discussion and open problems

In this section, we discuss relations to the literature, possible extensions and open problems.

The Ising model on random trees and random graphs. A key idea to analyze the Ising model on random graphs is to use the fact that expectations of local quantities coincide with the corresponding values for the Ising model on suitable random trees [11]. Statistical mechanics models on deterministic trees have been studied extensively in the literature (see for instance [2, 27] and its relation to "broadcasting on trees" in [19, 30]). The analysis on random trees is more recent and has been triggered by the study of models on random graphs. Extensions beyond the Ising model, e.g., the Potts model, pose new challenges [12].

Relation to the physics literature. Theorem 2.8 confirms the predictions in [16, 26]. For $\tau \leq 3$, one has $\nu = \infty$ and hence $\beta_c = 0$ by Theorem 2.7, so that the critical behavior coincides with the infinite temperature limit. Since in this case there is no phase transition at finite temperature, we do not study the critical behavior here. For $\tau = 5$, in [16], also the logarithmic correction for $\boldsymbol{\beta} = 1/2$ in (2.20) is computed, but not that of $\boldsymbol{\delta} = 3$.

The critical exponents $\boldsymbol{\gamma'}$ and other critical exponents. Theorem 2.8 only gives a lower bound on the critical exponent $\boldsymbol{\gamma'}$. It is predicted that $\boldsymbol{\gamma'} = 1$ for all $\tau > 3$, while there are also predictions for other critical exponents. For instance the critical exponent $\boldsymbol{\alpha'}$ for the specific heat in the low-temperature phase satisfies $\boldsymbol{\alpha'} = 0$ when $\mathbb{E}[K^3] < \infty$ and $\boldsymbol{\alpha'} = (\tau - 5)/(\tau - 3)$ in the power-law case with $\tau \in (3, 5)$ (see [16, 26]). We prove the lower bound $\boldsymbol{\gamma'} \geq 1$ in Section 6.2 below, and we also present a heuristic argument that $\boldsymbol{\gamma'} \leq 1$ holds. The critical exponent $\boldsymbol{\alpha'}$ for the specific heat is beyond our current methods, partly since we are not able to relate the specific heat on a random graph to that on the random Bethe tree.

Points of non-analyticity of the free energy. In Definition 2.2 we have defined the critical temperature as the highest temperature where the spontaneous magnetization is non-zero. This immediately implies that the free energy (proportional to the logarithm of the partition function defined in (2.2)) is non-analytic for $B = 0$ in the low-temperature phase. Indeed for $\beta > \beta_c$, $M(\beta, 0^+) > M(\beta, 0^-) = -M(\beta, 0^+)$, so that the first derivative of free energy with respect to the external field B has a jump crossing the line $B = 0$. The question arises if the phase diagram has more points of non-analyticity. The Lee-Yang Theorem [25] tells us that the free energy is an analytic function of $B \neq 0$. For the analyticity of the free energy as a function of the temperature we are not aware of *general* results that allow us to locate the zeros of the partition function (so-called "Fisher zeros" in the complex temperature plane). Although there are many studies on ferromagnets (see e.g., [6, 7]), the problem is largely open and model-dependent. For Ising and Potts models, a non-rigorous result [14] suggests that Fisher zeros on random trees and graphs should coincide.

Light tails. The case $\mathbb{E}[K^3] < \infty$ includes all power-law degree distributions with $\tau > 5$, but also cases where P does *not* obey a power law. This means, e.g., that Theorem 2.8 also identifies the critical exponents for the Erdős-Rényi random graph where the degrees have an asymptotic Poisson distribution.

Inclusion of slowly varying functions. In Definition 2.6, we have assumed that the asymptotic degree distribution obeys a perfect power law. Alternatively, one could assume that $\sum_{\ell \geq k} p_\ell \asymp L(k) k^{-(\tau-1)}$ for some function $k \mapsto L(k)$ that is slowly varying at $k = \infty$. For $\tau > 5$ and any slowly varying function, we still have $\mathbb{E}[K^3] < \infty$, so the results do not change and Theorem 2.8 remains to hold. For $\tau \in (3, 5]$, we expect slowly varying corrections to the critical behavior in Theorem 2.8. For example, $\mathbb{E}[K^3] < \infty$ for $\tau = 5$ and $L(k) = (\log k)^{-2}$, so that the logarithmic corrections present for $\tau = 5$ disappear.

Beyond the root magnetization for the random Bethe tree. We have identified the critical value and some critical exponents for the root magnetization on the random Bethe tree. The random Bethe tree is a so-called *unimodular* graph, which is a rooted graph that often arises as the local weak limit of a sequence of graphs (in this case, the random graphs $(G_n)_{n\geq 1}$). See [1, 4] for more background on unimodular graphs and trees, in particular, $\mathcal{T}(D, K, \infty)$ is the so-called *unimodular Galton-Watson tree* as proved by Lyons, Pemantle and Peres in [29]. One would expect that the magnetization of the graph, which can be defined by

$$M_T(\beta, B) = \lim_{t\to\infty} \frac{1}{|B_\phi(t)|} \sum_{v\in B_\phi(t)} \sigma_v, \tag{2.23}$$

where $B_\phi(t)$ is the graph induced by vertices at graph distance at most t from the root ϕ and $|B_\phi(t)|$ is the number of elements in it, also converges a.s. to a limit. However, we expect that $M_T(\beta, B) \neq M(\beta, B)$ due to the special role of the root ϕ, which vanishes in the above limit. Thus one would expect to believe that $M_T(\beta, B)$ equals the root magnetization of the tree where each vertex has degree distribution $K + 1$. Our results show that also $M_T(\beta, B)$ has the same critical temperature and critical exponents as $M(\beta, B)$.

Relation to the Curie-Weiss model. Our results show that locally tree-like random graphs with finite fourth moment of the degree distribution are in the same universality class as the mean-field model on the complete graph, which is the Curie-Weiss model. We further believe that the Curie-Weiss model should enter as the limit of $r \to \infty$ for the r-regular random graph, in the sense that these have the same critical exponents (as we already know), as well as that all constants arising in asymptotics match up nicely (cf. the discussion at the end of Section 6.2). Further, our results show that for $\tau \in (3, 5]$, the Ising model has *different* critical exponents than the ones for the Curie-Weiss model, so these constitute a set of different universality classes.

Organization of the article. The remainder of this article is organized as follows. We start with some preliminary computations in Section 3. In Section 4 we prove that the critical temperature is as stated in Theorem 2.7. The proof that the exponents stated in Theorem 2.8 are indeed the correct values of $\boldsymbol{\beta}$ and $\boldsymbol{\delta}$ is given in Section 5.3. The value of $\boldsymbol{\gamma}$ is identified in Section 6, where also the lower bound on $\boldsymbol{\gamma'}$ is proved and a heuristic is presented for the matching upper bound.

3 Preliminaries

An important role in our analysis is played by the distributional recursion

$$h^{(t+1)} \stackrel{d}{=} B + \sum_{i=1}^{K_t} \xi(h_i^{(t)}), \tag{3.1}$$

where

$$\xi(h) = \operatorname{atanh}(\theta \tanh(h)), \tag{3.2}$$

with $\theta = \tanh(\beta)$, and where $h^{(0)} \equiv B$, $(K_t)_{t\geq 1}$, are i.i.d. with distribution ρ and $(h_i^{(t)})_{i\geq 1}$ are i.i.d. copies of $h^{(t)}$ independent of K_t. In [15, Proposition 1.7], we have proved that this recursion has a unique distributional fixed point h that is supported on $[0, \infty)$ for all $\beta \geq 0$ and $B > 0$. Whenever we write h or h_i this is a random variable distributed as the fixed point of (3.1). Since h is a fixed point, we can interchange $h \stackrel{\mathrm{d}}{=} B + \sum_{i=1}^{K} \xi(h_i)$ in expectations and we often do this. We also often use the facts that $h \geq 0$ for $B \geq 0$ (with equality iff $B = 0$) and $0 \leq \xi(h) \leq \beta$ for $h \geq 0$.

This fixed point h yields the random field acting on the root of the random Bethe tree $\mathcal{T}(D, K, \infty)$ due to its offsprings. In particular we can use the fixed point h to give an explicit expression for the magnetization:

Proposition 3.1 (Magnetization)**.** *Assume that the random graph sequence $(G_n)_{n\geq 1}$ is locally tree-like with asymptotic degree distribution P that obeys $\mathbb{E}[K] < \infty$ or a power law with exponent $\tau > 2$ and is uniformly sparse. Then, a.s., for all $\beta \geq 0$ and $B > 0$, the thermodynamic limit of the* magnetization *per vertex exists and is given by*

$$M(\beta, B) = \mathbb{E}\Big[\tanh\Big(B + \sum_{i=1}^{D} \xi(h_i)\Big)\Big], \tag{3.3}$$

where

(i) *D has distribution P;*

(ii) *$(h_i)_{i\geq 1}$ are i.i.d. copies of the fixed point of the distributional recursion* (3.1)*;*

(iii) *D and $(h_i)_{i\geq 1}$ are independent.*

The same holds on the random Bethe tree $\mathcal{T}(D, K, \infty)$.

This proposition was proved in [15, Corollary 1.6(a)] by differentiating the expression for the thermodynamic limit of the pressure per particle that was first obtained. Here we present a more intuitive proof:

Proof of Proposition 3.1. Let ϕ be a vertex picked uniformly at random from $[n]$ and $\mathbb{E}_n$ be the corresponding expectation. Then,

$$M_n(\beta, B) = \frac{1}{n} \sum_{i=1}^{n} \langle \sigma_i \rangle_{\mu_n} = \mathbb{E}_n[\langle \sigma_\phi \rangle_{\mu_n}]. \tag{3.4}$$

Denote by $\langle \cdot \rangle_{\mu_n}^{\ell,+/f}$ the expectations with respect to the Ising measure with $+$/free boundary conditions on vertices at graph distance ℓ from ϕ. Note that $\langle \sigma_\phi \rangle_{\mu_n}^{\ell,+/f}$ only depends on the spins of vertices in $B_\phi(\ell)$. By the GKS inequality [24],

$$\langle \sigma_\phi \rangle_{\mu_n}^{\ell,f} \leq \langle \sigma_\phi \rangle_{\mu_n} \leq \langle \sigma_\phi \rangle_{\mu_n}^{\ell,+}. \tag{3.5}$$

Taking the limit $n \to \infty$, the ball $B_\phi(\ell)$ has the same distribution as the random tree $\mathcal{T}(D, K, \ell)$, because of the locally tree-like nature of the graph sequence. With $\langle \cdot \rangle^{\ell,+/f}$ denoting the expectations with respect to the Ising measure with $+$/free boundary conditions on $\mathcal{T}(D, K, \ell)$, this means that

$$\lim_{n\to\infty} \langle \sigma_\phi \rangle_{\mu_n}^{\ell,+/f} = \langle \sigma_\phi \rangle^{\ell,+/f}. \tag{3.6}$$

Conditioned on the tree $\mathcal{T}$, we can prune the tree, see [11, Lemma 4.1], to obtain that

$$\langle\sigma_\phi\rangle^{\ell,f} = \tanh\Big(B + \sum_{i=1}^{D}\xi(h_i^{(\ell-1)})\Big). \tag{3.7}$$

Similarly,

$$\langle\sigma_\phi\rangle^{\ell,+} = \tanh\Big(B + \sum_{i=1}^{D}\xi(h_i^{'(\ell-1)})\Big), \tag{3.8}$$

where $h_i^{'(t+1)}$ also satisfies (3.1), but has initial value $h^{'(0)} = \infty$. Since this recursion has a unique fixed point [15, Proposition 1.7], we prove the proposition by taking the limit $\ell \to \infty$ and taking the expectation over the tree $\mathcal{T}(D, K, \infty)$. □

To study the critical behavior we investigate the function $\xi(x) = \operatorname{atanh}(\theta \tanh x)$ and prove two important bounds that play a crucial role throughout this paper:

Lemma 3.2 (Properties of $x \mapsto \xi(x)$)**.** *For all $x, \beta \geq 0$,*

$$\theta x - \frac{\theta}{3(1-\theta^2)}x^3 \leq \xi(x) \leq \theta x. \tag{3.9}$$

The upper bound holds with strict inequality if $x, \beta > 0$.

Proof. By Taylor's theorem,

$$\xi(x) = \xi(0) + \xi'(0)x + \xi''(\zeta)\frac{x^2}{2}, \tag{3.10}$$

for some $\zeta \in (0, x)$. It is easily verified that $\xi(0) = 0$,

$$\xi'(0) = \frac{\theta(1-\tanh^2 x)}{1-\theta^2\tanh^2 x}\bigg|_{x=0} = \theta, \tag{3.11}$$

and

$$\xi''(\zeta) = -\frac{2\theta(1-\theta^2)(\tanh\zeta)(1-\tanh^2\zeta)}{(1-\theta^2\tanh^2\zeta)^2} \leq 0, \tag{3.12}$$

thus proving the upper bound. If $x, \beta > 0$ then also $\zeta > 0$ and hence the above holds with strict inequality.

For the lower bound, note that $\xi''(0) = 0$ and

$$\begin{aligned} \xi'''(\zeta) &= -\frac{2\theta(1-\theta^2)(1-\tanh^2\zeta)}{(1-\theta^2\tanh^2\zeta)^3}\left(1 - 3(1-\theta^2)\tanh^2\zeta - \theta^2\tanh^4\zeta\right) \\ &\geq -\frac{2\theta(1-\theta^2)(1-\tanh^2\zeta)}{(1-\theta^2)^2(1-\tanh^2\zeta)} = -\frac{2\theta}{1-\theta^2}. \end{aligned} \tag{3.13}$$

Thus, for some $\zeta \in (0, x)$,

$$\xi(x) = \xi(0) + \xi'(0)x + \xi''(0)\frac{x^2}{2} + \xi'''(\zeta)\frac{x^3}{3!} \geq \theta x - \frac{2\theta}{1-\theta^2}\frac{x^3}{3!}. \tag{3.14}$$

□

We next study tail probabilites of $(\rho_k)_{k\geq 0}$. Here, for a probability distribution $(q_k)_{k\geq 0}$ on the integers, we write $q_{\geq k} = \sum_{\ell\geq k} q_\ell$.

Lemma 3.3 (Tail probabilities of $(\rho_k)_{k\geq 0}$)**.** *Assume that* (2.15) *holds for some* $\tau > 2$. *Then, for the size-biased distribution defined in (2.12), there exist* $0 < c_\rho \leq C_\rho$ *such that, for all* $k \geq 1$,

$$c_\rho k^{-(\tau-2)} \leq \rho_{\geq k} \leq C_\rho k^{-(\tau-2)}. \tag{3.15}$$

Proof. The lower bound follows directly from the fact that $\rho_{\geq k} \geq (k+1)p_{\geq k+1}/\mathbb{E}[D]$, and (2.15). For the upper bound, we note that for any probability distribution $(q_k)_{k\geq 0}$ on the non-negative integers, we have the partial summation identity

$$\sum_{k\geq 0} q_k f(k) = f(0) + \sum_{\ell\geq 1} q_{\geq \ell}[f(\ell) - f(\ell-1)], \tag{3.16}$$

provided that either $[f(\ell)-f(\ell-1)]q_{\geq\ell}$ is absolutely summable, or $k \mapsto f(k)$ is either non-decreasing or non-increasing. Indeed,

$$\sum_{k\geq 0} q_k f(k) = f(0) + \sum_{k=0}^{\infty} q_k[f(k) - f(0)] = f(0) + \sum_{k=0}^{\infty} q_k \sum_{\ell=1}^{k}[f(\ell) - f(\ell-1)], \tag{3.17}$$

and the claim follows by interchanging the summation order, which is allowed by Fubini's Theorem for non-negative functions (see [21, Section 3.6, Theorem B]) when $k \mapsto f(k)$ is non-decreasing, and by Fubini's Theorem [21, Section 3.6, Theorem C] when $[f(\ell)-f(\ell-1)]\mathbb{1}_{\{0\leq\ell\leq k\}}q_k$ is absolutely summable, which, by non-negativity of q_k, is equivalent to the absolutely summability of $[f(\ell) - f(\ell-1)]q_{\geq\ell}$.

We start by proving bounds on $\rho_{\geq k}$. We rewrite

$$\rho_{\geq k} = \sum_{\ell\geq k} \frac{(\ell+1)p_{\ell+1}}{\mathbb{E}[D]} = \sum_{\ell\geq 0} f(\ell)p_{\ell+1}, \tag{3.18}$$

where $f(\ell) = (\ell+1)\mathbb{1}_{\{\ell\geq k\}}/\mathbb{E}[D]$. By (3.16) with $q_\ell = p_{\ell+1}$, for $k \geq 1$ so that $f(0) = 0$,

$$\rho_{\geq k} = \sum_{\ell\geq 1}[f(\ell) - f(\ell-1)]p_{\geq \ell+1} = \frac{(k+1)p_{\geq k+1}}{\mathbb{E}[D]} + \frac{1}{\mathbb{E}[D]} \sum_{\ell\geq k+1} p_{\geq \ell+1}. \tag{3.19}$$

From (2.15), it follows that

$$\rho_{\geq k} \leq \frac{C_p}{\mathbb{E}[D]}(k+1)^{-(\tau-2)} + \sum_{\ell\geq k+1} \frac{C_p}{\mathbb{E}[D]}(\ell+1)^{-(\tau-1)}, \tag{3.20}$$

so that there exists a constant C_ρ such that

$$\rho_{\geq k} \leq C_\rho k^{-(\tau-2)}. \tag{3.21}$$

□

When computing the critical exponents for $\tau \in (3,5]$, we often split the analysis into two cases: one where K is small and one where K is large. For this we need bounds on truncated moments of K which are the content of the next lemma.

Lemma 3.4 (Truncated moments of K)**.** *Assume that* (2.15) *holds for some* $\tau > 2$*. Then there exist constants* $C_{a,\tau} = C_{a,\tau}(C_\rho) > 0$ *such that, as* $\ell \to \infty$*,*

$$\mathbb{E}\left[K^a \mathbb{1}_{\{K\leq \ell\}}\right] \leq \begin{cases} C_{a,\tau}\ell^{a-(\tau-2)} & \text{when } a > \tau - 2, \\ C_{\tau-2,\tau}\log \ell & \text{when } a = \tau-2. \end{cases} \tag{3.22}$$

and, when $a < \tau - 2$*,*

$$\mathbb{E}\left[K^a \mathbb{1}_{\{K> \ell\}}\right] \leq C_{a,\tau}\ell^{a-(\tau-2)}. \tag{3.23}$$

Finally, when $\tau = 5$*, there exists a constant* $c_{3,5} = c_{3,5}(c_\rho) > 0$ *such that, as* $\ell \to \infty$*,*

$$\mathbb{E}\left[K(K-1)(K-2)\mathbb{1}_{\{K\leq \ell\}}\right] \geq c_{3,5}\log \ell. \tag{3.24}$$

Proof. We start by bounding the truncated moments of K. We rewrite, using (3.16) and with $f(k) = k^a \mathbb{1}_{\{k\leq \ell\}}$,

$$\mathbb{E}\left[K^a \mathbb{1}_{\{K\leq \ell\}}\right] = \sum_{k=0}^{\infty} f(k)\rho_k = \sum_{k=1}^{\infty}[f(k)-f(k-1)]\rho_{\geq k} \leq \sum_{k=1}^{\lfloor \ell \rfloor}[k^a-(k-1)^a]\rho_{\geq k}. \tag{3.25}$$

Using $k^a-(k-1)^a = a\int_{k-1}^k x^{a-1}dx \leq ak^{a-1}$, we arrive at

$$\mathbb{E}\left[K^a \mathbb{1}_{\{K\leq \ell\}}\right] \leq aC_\rho \sum_{k=1}^{\lfloor \ell \rfloor} k^{a-1}k^{-(\tau-2)} \leq aC_\rho \sum_{k=1}^{\lfloor \ell \rfloor+1} k^{a-(\tau-1)}. \tag{3.26}$$

Note that $k \mapsto k^{a-(\tau-1)}$ is either increasing or decreasing. Hence,

$$\sum_{k=1}^{\lfloor \ell \rfloor+1} k^{a-(\tau-1)} \leq \int_1^{\ell+2} k^{a-(\tau-1)}\mathrm{d}k. \tag{3.27}$$

For $a > \tau - 2$,

$$\int_1^{\ell+2} k^{a-(\tau-1)}\mathrm{d}k \leq \frac{2}{a+2-\tau}\ell^{a-(\tau-2)}, \tag{3.28}$$

whereas for $a = \tau - 2$,

$$\int_1^{\ell+2} k^{a-(\tau-1)}\mathrm{d}k \leq 2\log \ell. \tag{3.29}$$

Similarly, for $a < \tau - 2$,

$$\begin{aligned} \mathbb{E}\left[K^a \mathbb{1}_{\{K> \ell\}}\right] &= \lceil \ell \rceil^a \rho_{\geq \ell} + \sum_{k>\ell}[k^a-(k-1)^a]\rho_{\geq k} \\ &\leq C_\rho \lceil \ell \rceil^{a-(\tau-2)} + aC_\rho \sum_{\lfloor \ell \rfloor+1}^{\infty} k^{a-1}(k+1)^{-(\tau-2)} \leq C_{a,\tau}\ell^{a-(\tau-2)}. \end{aligned} \tag{3.30}$$

Finally, we prove (3.24), for which we compute with $f(k) = k(k-1)(k-2)$,

$$\mathbb{E}\left[K(K-1)(K-2)\mathbb{1}_{\{K\leq\ell\}}\right] = \sum_{k=1}^{\infty}[f(k)-f(k-1)]\sum_{l=k}^{\ell}\rho_l = \sum_{k=3}^{\infty}3(k-1)(k-2)\sum_{l=k}^{\ell}\rho_l. \quad (3.31)$$

We bound this from below by

$$\mathbb{E}\left[K(K-1)(K-2)\mathbb{1}_{\{K\leq\ell\}}\right] \geq \sum_{k=0}^{\sqrt{\ell}}3(k-1)(k-2)[\rho_{\geq k}-\rho_{\geq \ell}]. \quad (3.32)$$

By Lemma 3.3, for $\tau = 5$, the contribution due to $\rho_{\geq\ell}$ is at most

$$\ell^{3/2}\rho_{\geq\ell} \leq C_\rho \ell^{-3/2} = o(1), \quad (3.33)$$

while the contribution due to $\rho_{\geq k}$ and using $3(k-1)(k-2) \geq k^2$ for every $k \geq 4$, is at least

$$c_\rho\sum_{k=4}^{\sqrt{\ell}}k^{-1} \geq c_\rho\int_4^{\sqrt{\ell}+1}\frac{\mathrm{d}x}{x} = c_\rho[\log(\sqrt{\ell}+1)-\log 4], \quad (3.34)$$

which proves the claim by chosing the constant $c_{3,5}$ correctly. □

4 Critical temperature

In this section we compute the critical temperature.

Proof of Theorem 2.7. Let $\beta^* = \operatorname{atanh}(1/\nu)$. We first show that if $\beta < \beta^*$, then

$$\lim_{B\searrow 0} M(\beta, B) = 0, \quad (4.1)$$

which implies that $\beta_c \geq \beta^*$. Later, we show that if $\lim_{B\searrow 0} M(\beta,B) = 0$ then $\beta \leq \beta^*$, implying that $\beta_c \leq \beta^*$.

Proof of $\beta_c \geq \beta^*$. Suppose that $\beta < \beta^*$. Then, by the fact that $\tanh x \leq x$ and Wald's identity,

$$M(\beta,B) = \mathbb{E}\left[\tanh\left(B + \sum_{i=1}^{D}\xi(h_i)\right)\right] \leq B + \mathbb{E}[D]\mathbb{E}[\xi(h)]. \quad (4.2)$$

We use the upper bound in Lemma 3.2 to get

$$\mathbb{E}[\xi(h)] = \mathbb{E}[\operatorname{atanh}(\theta\tanh h)] \leq \theta\mathbb{E}[h] = \theta\left(B + \nu\mathbb{E}[\xi(h)]\right). \quad (4.3)$$

Further, note that

$$\mathbb{E}[\xi(h)] = \mathbb{E}[\operatorname{atanh}(\theta\tanh h)] \leq \beta, \quad (4.4)$$

because $\tanh h \leq 1$. Applying inequality (4.3) ℓ times to (4.2) and subsequently using inequality (4.4) once gives

$$M(\beta, B) \leq B + B\theta\mathbb{E}[D]\frac{1-(\theta\nu)^\ell}{1-\theta\nu} + \beta\mathbb{E}[D](\theta\nu)^\ell. \tag{4.5}$$

Hence,

$$\begin{aligned} M(\beta, B) &\leq \limsup_{\ell\to\infty}\left(B + B\theta\mathbb{E}[D]\frac{1-(\theta\nu)^\ell}{1-\theta\nu} + \beta\mathbb{E}[D](\theta\nu)^\ell\right) \\ &= B\left(1+\theta\mathbb{E}[D]\frac{1}{1-\theta\nu}\right), \end{aligned} \tag{4.6}$$

because $\theta < \theta^* = 1/\nu$. Therefore,

$$\lim_{B\searrow 0} M(\beta, B) \leq \lim_{B\searrow 0} B\left(1+\theta\mathbb{E}[D]\frac{1}{1-\theta\nu}\right) = 0. \tag{4.7}$$

This proves the lower bound on β_c.

Proof of $\beta_c \leq \beta^*$. We adapt Lyons' proof in [27] for the critical temperature of deterministic trees to the random tree to show that $\beta_c \leq \beta^*$. Assume that $\lim_{B\searrow 0} M(\beta, B) = 0$. Note that Proposition 3.1 shows that the magnetization $M(\beta, B)$ is equal to the expectation over the random tree $\mathcal{T}(D, K, \infty)$ of the root magnetization. Hence, if we denote the root of the tree $\mathcal{T}(D, K, \infty)$ by ϕ, then $M(\beta, B) = \mathbb{E}[\langle\sigma_\phi\rangle]$. It follows from our assumption on $M(\beta, B)$ that, a.s., $\lim_{B\searrow 0}\langle\sigma_\phi\rangle = 0$, since the latter limit exists by the GKS inequalities.

We therefore condition on the tree $T = \mathcal{T}(D, K, \infty)$. Define for $v \in T$

$$h(v) = \langle\sigma_v\rangle \qquad \text{and} \qquad h^{\ell,+}(v) = \langle\sigma_v\rangle^{\ell,+}, \tag{4.8}$$

and let $|v|$ denote the graph distance from ϕ to v. Furthermore, we say that $w \leftarrow v$ if $\{w, v\}$ is an edge in T and $|w| = |v| + 1$. By [11, Lemma 4.1], for $|v| < \ell$,

$$h^{\ell,+}(v) = B + \sum_{w\leftarrow v}\xi(h^{\ell,+}(w)). \tag{4.9}$$

Since this recursion has a unique solution by [15, Proposition 1.7] we have $h(\phi) = \lim_{\ell\to\infty} h^{\ell,+}(\phi)$. Therefore, if we suppose that $\lim_{B\searrow 0}\langle\sigma_\phi\rangle = 0$, then also $\lim_{B\searrow 0} h(\phi) = 0$ and then it thus also holds that $\lim_{B\searrow 0}\lim_{\ell\to\infty} h^{\ell,+}(\phi) = 0$. Because of (4.9), we must then have, for all $v \in T$,

$$\lim_{B\searrow 0}\lim_{\ell\to\infty} h^{\ell,+}(v) = 0. \tag{4.10}$$

Now, fix $0 < \beta_0 < \beta$ and choose ℓ large enough and B small enough such that, for some $\varepsilon = \varepsilon(\beta_0, \beta) > 0$ that we choose later,

$$h^{\ell,+}(v) \leq \varepsilon, \tag{4.11}$$

for all $v \in T$ with $|v| = 1$. Note that $h^{\ell,+}(v) = \infty > \varepsilon$ for $v \in T$ with $|v| = \ell$.

As in [27], we say that Π is a *cutset* if Π is a finite subset of $T \setminus \{\phi\}$ and every path from ϕ to infinity intersects Π at exactly one vertex $v \in \Pi$. We write $v \leq \Pi$ if every infinite path from v

intersects Π and write $\sigma < \Pi$ if $\sigma \leq \Pi$ and $\sigma \notin \Pi$. Then, since $h^{\ell,+}(v) \leq \varepsilon$ for $v \in \mathcal{T}$ with $|v| = 1$ and $h^{\ell,+}(v) = \infty > \varepsilon$ for $v \in T$ with $|v| = \ell$, there is a unique cutset Π_ℓ, such that $h^{\ell,+}(v) \leq \varepsilon$ for all $v \leq \Pi_\ell$, and for all $v \in \Pi_\ell$ there is at least one $w \leftarrow v$ such that $h^{\ell,+}(w) > \varepsilon$.

It follows from the lower bound in Lemma 3.2 that, for $v < \Pi_\ell$,

$$h^{\ell,+}(v) = B + \sum_{w \leftarrow v} \xi(h^{\ell,+}(w)) \geq \sum_{w \leftarrow v} \theta h^{\ell,+}(w) - \frac{\theta h^{\ell,+}(w)^3}{3(1-\theta^2)} \geq \sum_{w \leftarrow v} \theta h^{\ell,+}(w)\Big(1 - \frac{\varepsilon^2}{3(1-\theta^2)}\Big), \quad (4.12)$$

while, for $v \in \Pi_\ell$,

$$h^{\ell,+}(v) = B + \sum_{w \leftarrow v} \xi(h^{\ell,+}(w)) > \xi(\varepsilon). \quad (4.13)$$

If we now choose $\varepsilon > 0$ such that

$$\theta\Big(1 - \frac{\varepsilon^2}{3(1-\theta^2)^2}\Big) = \theta_0, \quad (4.14)$$

which is possible because $\beta_0 < \beta$, then, iterating (4.12) in each direction until Π_ℓ and then using (4.13),

$$h^{\ell,+}(\phi) \geq \sum_{v \in \Pi_\ell} \theta_0^{|v|} \xi(\varepsilon). \quad (4.15)$$

Since $\xi(\varepsilon) > 0$ and $\lim_{B \searrow 0} \lim_{\ell \to \infty} h^{\ell,+}(\phi) = 0$,

$$\inf_{\Pi} \sum_{v \in \Pi} \theta_0^{|v|} = 0. \quad (4.16)$$

From [28, Proposition 6.4] it follows that $\theta_0 \leq 1/\nu$. This holds for all $\beta_0 < \beta$, so

$$\beta \leq \operatorname{atanh}(1/\nu) = \beta^*. \quad (4.17)$$

This proves the upper bound on β_c, thus concluding the proof. □

We next show that the phase transition at this critical temperature is *continuous:*

Lemma 4.1 (Continuous phase transition)**.** *Let $((\beta_n, B_n))_{n\geq1}$ be a sequence with β_n and B_n non-increasing, $\beta_n \geq \beta_c$ and $B_n > 0$, and $\beta_n \searrow \beta_c$ and $B_n \searrow 0$ as $n \to \infty$. Then,*

$$\lim_{n\to\infty} \mathbb{E}[\xi(\beta_n, B_n)] = 0. \quad (4.18)$$

In particular,

$$\lim_{B \searrow 0} \mathbb{E}[\xi(h(\beta_c, B))] = 0, \qquad \text{and} \qquad \lim_{\beta \searrow \beta_c} \mathbb{E}[\xi(h(\beta, 0^+))] = 0. \quad (4.19)$$

Proof. For all sequences $((\beta_n, B_n))_{n\geq1}$ satisfying the assumptions stated in the lemma, $\mathbb{E}[\xi(\beta_n, B_n)]$ is non-increasing in n and it is also non-negative so that the limit as $n \to \infty$ exists. By the concavity of $h \mapsto \xi(h)$ and Jensen's inequality,

$$0 \leq c \equiv \lim_{n\to\infty} \mathbb{E}[\xi(\beta_n, B_n)] \leq \lim_{n\to\infty} \xi\big(B_n + \nu\mathbb{E}[\xi(h(\beta_n, B_n))]\big) = \xi(\nu c). \quad (4.20)$$

Since $\xi(x) < \theta_c x$ for $x > 0$ by Lemma 3.2 and using $\theta_c = 1/\nu$, we obtain

$$\xi(\nu c) < \theta_c \nu c = c, \quad (4.21)$$

leading to a contradiction when $c > 0$. □

5 Critical exponents: Magnetization

In this section we prove that the critical exponents related to the magnetization, i.e., $\boldsymbol{\beta}$ and $\boldsymbol{\delta}$, take the values stated in Theorem 2.8. The analysis involves Taylor expansions performed up to the right order. By these Taylor expansions, higher moments of $\xi(h)$ appear, where we remind the reader that $\xi(h) = \operatorname{atanh}(\theta \tanh h)$ and h is the unique non-negative distributional fixed point of (3.1). Therefore, we first bound these higher moments of $\xi(h)$ in terms of its first moment in Section 5.1.

In Section 5.2 we use these bounds to give appropriate bounds on $\mathbb{E}[\xi(h)]$ which finally allow us to compute the critical exponents $\boldsymbol{\beta}$ and $\boldsymbol{\delta}$ in Section 5.3.

Throughout Section 5 we assume that B is sufficiently close to zero and $\beta_c < \beta < \beta_c + \varepsilon$ for ε sufficiently small, so that we can apply Lemma 4.1 to make sure $\mathbb{E}[\xi(h)]$ is sufficiently small.

In the following, we write $c_i, C_i, i \geq 1$ for constants that have the properties: i) they only depend on β and moments of K (they do not depend on B); ii) they satisfy

$$0 < \liminf_{\beta \searrow \beta_c} c_i(\beta) \leq \limsup_{\beta \searrow \beta_c} c_i(\beta) < \infty, \tag{5.1}$$

and the same holds for the C_i. The index i is just a label for the constants. For reader convenience we try to use the i^{th} label to denote a constant appearing in a bound involving the i^{th} moment of $\xi(h)$. However this is not always possible and therefore in general nothing should be sought from the labeling of the constants. What we consistently do is to use C_i for constants appearing in upper bounds, while c_i appears in lower bounds. Furthermore, we write $e_i, i \geq 1$ (again the labeling is arbitrary) for error functions that only depend on $\beta, B, \mathbb{E}[\xi(h)]$ and moments of K, and satisfy for all $\beta \in (\beta_c, \beta_c + \epsilon)$

$$\limsup_{B \searrow 0} e_i(\beta, B) < \infty \qquad \text{and} \qquad \lim_{B \searrow 0} e_i(\beta_c, B) = 0. \tag{5.2}$$

The expressions of c_i, C_i, e_i are given in the proofs of the results of this section and conditions (5.1) (5.2) are verified explicitly . Finally, we write $\nu_k = \mathbb{E}[K(K-1)\cdots(K-k+1)]$ for the kth factorial moment of K, so that $\nu_1 = \nu$.

5.1 Bounds on higher moments of $\xi(h)$

We start with bounding the second moment of $\xi(h)$.

Lemma 5.1 (Bounds on second moment of $\xi(h)$)**.** *Let $\beta \geq \beta_c$ and $B > 0$. Then,*

$$\mathbb{E}[\xi(h)^2] \leq \begin{cases} C_2 \mathbb{E}[\xi(h)]^2 + Be_2 & \text{when } \mathbb{E}[K^2] < \infty, \\ C_2 \mathbb{E}[\xi(h)]^2 \log\left(1/\mathbb{E}[\xi(h)]\right) + Be_2 & \text{when } \tau = 4, \\ C_2 \mathbb{E}[\xi(h)]^{\tau-2} + Be_2 & \text{when } \tau \in (3,4). \end{cases} \tag{5.3}$$

Proof. We first treat the case $\mathbb{E}[K^2] < \infty$. We use Lemma 3.2 and the recursion in (3.1) to obtain

$$\begin{aligned} \mathbb{E}[\xi(h)^2] &\leq \theta^2 \mathbb{E}[h^2] = \theta^2 \mathbb{E}\Big[\Big(B + \sum_{i=1}^{K} \xi(h_i)\Big)^2\Big] \\ &= \theta^2 \left(B^2 + 2B\nu \mathbb{E}[\xi(h)] + \nu_2 \mathbb{E}[\xi(h)]^2 + \nu \mathbb{E}[\xi(h)^2]\right). \end{aligned} \tag{5.4}$$

Since $1-\theta^2\nu > 0$, because β is sufficiently close to β_c and $\theta_c = 1/\nu < 1$, the lemma holds with

$$C_2 = \frac{\theta^2\nu_2}{1-\theta^2\nu}, \qquad \text{and} \qquad e_2 = \frac{B\theta^2 + 2\theta^2\nu\mathbb{E}[\xi(h)]}{1-\theta^2\nu}. \tag{5.5}$$

It is not hard to see that (5.1) holds. For e_2 the first property of (5.2) can also easily be seen. The second property in (5.2) follows from Lemma 4.1.

If $\tau \le 4$, then $\mathbb{E}[K^2] = \infty$ and the above does not work. To analyze this case, we apply the recursion (3.1) and split the expectation over K in small and large degrees:

$$\mathbb{E}[\xi(h)^2] = \mathbb{E}\Big[\xi\Big(B + \sum_{i=1}^{K}\xi(h_i)\Big)^2 \mathbb{1}_{\{K\le\ell\}}\Big] + \mathbb{E}\Big[\xi\Big(B + \sum_{i=1}^{K}\xi(h_i)\Big)^2 \mathbb{1}_{\{K>\ell\}}\Big]. \tag{5.6}$$

We use Lemma 3.2 to bound the first term as follows:

$$\begin{aligned}\mathbb{E}\Big[\xi\Big(B + \sum_{i=1}^{K}\xi(h_i)\Big)^2 \mathbb{1}_{\{K\le\ell\}}\Big] &\le \theta^2\mathbb{E}\Big[\Big(B + \sum_{i=1}^{K}\xi(h_i)\Big)^2 \mathbb{1}_{\{K\le\ell\}}\Big] \\ &\le \theta^2\left(B^2 + 2B\nu\mathbb{E}[\xi(h)] + \mathbb{E}[K^2\mathbb{1}_{\{K\le\ell\}}]\mathbb{E}[\xi(h)]^2 + \nu\mathbb{E}[\xi(h)^2]\right),\end{aligned} \tag{5.7}$$

where in the second inequality we used the indepence of the $\xi(h_i)$ and Wald's identity, and bounded $\mathbb{E}[K\mathbb{1}_{\{K\le\ell\}}] \le \nu$. For $\tau \in (3,4)$,

$$\mathbb{E}[K^2\mathbb{1}_{\{K\le\ell\}}] \le C_{2,\tau}\ell^{4-\tau}, \tag{5.8}$$

by Lemma 3.4, while for $\tau = 4$,

$$\mathbb{E}[K^2\mathbb{1}_{\{K\le\ell\}}] \le C_{2,4}\log\ell. \tag{5.9}$$

To bound the second sum in (5.6), note that $\xi(h) \le \beta$. Hence,

$$\mathbb{E}\Big[\xi\Big(B + \sum_{i=1}^{K}\xi(h_i)\Big)^2 \mathbb{1}_{\{K>\ell\}}\Big] \le \beta^2\mathbb{E}[\mathbb{1}_{\{K>\ell\}}] \le C_{0,\tau}\beta^2\ell^{2-\tau}. \tag{5.10}$$

The optimal bound (up to a constant) can be achieved by choosing ℓ such that $\ell^{4-\tau}\mathbb{E}[\xi(h)]^2$ and $\ell^{2-\tau}$ are of the same order of magnitude. Hence, we choose $\ell = 1/\mathbb{E}[\xi(h)]$. This choice of the truncation value ℓ turns out to be the relevant choice when making the distinction between small and large values of K also in further computations and hence will also be used (up to a constant) there.

Combining the two upper bounds gives the desired result with

$$C_2 = \frac{1}{1-\theta^2\nu}\left(C_{2,\tau}\theta^2 + C_{0,\tau}\beta^2\right), \tag{5.11}$$

where, for $\tau = 4$, we have also used that $\mathbb{E}[\xi(h)]^2 \le \mathbb{E}[\xi(h)]^2\log(1/\mathbb{E}[\xi(h)])$, and

$$e_2 = \frac{B\theta^2 + 2\theta^2\nu\mathbb{E}[\xi(h)]}{1-\theta^2\nu}. \tag{5.12}$$

□

We next derive upper bounds on the third moment of $\xi(h)$:

Lemma 5.2 (Bounds on third moment of $\xi(h)$)**.** *Let $\beta \geq \beta_c$ and $B > 0$. Then,*

$$\mathbb{E}[\xi(h)^3] \leq \begin{cases} C_3\mathbb{E}[\xi(h)]^3 + Be_3 & \text{when } \mathbb{E}[K^3] < \infty, \\ C_3\mathbb{E}[\xi(h)]^3 \log\left(1/\mathbb{E}[\xi(h)]\right) + Be_3 & \text{when } \tau = 5, \\ C_3\mathbb{E}[\xi(h)]^{\tau-2} + Be_3 & \text{when } \tau \in (3,5). \end{cases} \tag{5.13}$$

Proof. For $\mathbb{E}[K^3] < \infty$ we bound, in a similar way as in Lemma 5.1,

$$\begin{aligned} \mathbb{E}[\xi(h)^3] \leq \theta^3 \Bigg(& B^3 + 3B^2\nu\mathbb{E}[\xi(h)] + 3B\nu_2\mathbb{E}[\xi(h)]^2 + 3B\nu\mathbb{E}[\xi(h)^2] \\ & + \nu_3\mathbb{E}[\xi(h)]^3 + 3\nu_2\mathbb{E}[\xi(h)]\mathbb{E}[\xi(h)^2] + \nu\mathbb{E}[\xi(h)^3]\Bigg). \end{aligned} \tag{5.14}$$

Using (5.3), we indeed get the bound

$$\mathbb{E}[\xi(h)^3] \leq C_3\mathbb{E}[\xi(h)]^3 + Be_3, \tag{5.15}$$

where

$$C_3 = \frac{\theta^3}{1-\theta^3\nu}\left(\nu_3 + 3\nu_2 C_2\right), \tag{5.16}$$

and

$$e_3 = \frac{\theta^3}{1-\theta^3\nu}\left\{B^2 + 3B\nu e_2 + 3\left(B\nu + \nu_2 e_2\right)\mathbb{E}[\xi(h)] + 3\left(\nu_2 + \nu C_2\right)\mathbb{E}[\xi(h)]^2\right\}. \tag{5.17}$$

To see that C_3 satisfies (5.1), note that $\nu_2, \nu_3 < \infty$ since $\mathbb{E}[K^3] < \infty$. Furthermore, $\nu_2 > 0$ since $\mathbb{P}[K \geq 2] > 0$ because $\nu > 1$ and $\nu_3 \geq 0$ because K can only take non-negative integer values. That e_3 satisfies (5.2) follows from the bound $\mathbb{E}[\xi(h)] \leq \beta < \infty$ and Lemma 4.1.

For $\tau \in (3,5]$, we use the recursion (3.1), make the expectation over K explicit and split in small and large values of K to obtain

$$\mathbb{E}[\xi(h)^3] = \mathbb{E}\Big[\xi\Big(B + \sum_{i=1}^{K}\xi(h_i)\Big)^3 \mathbb{1}_{\{K \leq 1/\mathbb{E}[\xi(h)]\}}\Big] + \mathbb{E}\Big[\xi\Big(B + \sum_{i=1}^{K}\xi(h_i)\Big)^3 \mathbb{1}_{\{K > 1/\mathbb{E}[\xi(h)]\}}\Big]. \tag{5.18}$$

We bound the first expectation from above by

$$\begin{aligned} & \theta^3\mathbb{E}\Big[\Big(B + \sum_{i=1}^{K}\xi(h_i)\Big)^3 \mathbb{1}_{\{K \leq 1/\mathbb{E}[\xi(h)]\}}\Big] \\ &\quad = \theta^3\Big(B^3 + 3B^2\mathbb{E}[K\mathbb{1}_{\{K \leq 1/\mathbb{E}[\xi(h)]\}}]\mathbb{E}[\xi(h)] + 3B\mathbb{E}[K(K-1)\mathbb{1}_{\{K \leq 1/\mathbb{E}[\xi(h)]\}}]\mathbb{E}[\xi(h)]^2 \\ &\qquad + 3B\mathbb{E}[K\mathbb{1}_{\{K \leq 1/\mathbb{E}[\xi(h)]\}}]\mathbb{E}[\xi(h)^2] + \mathbb{E}[K(K-1)(K-2)\mathbb{1}_{\{K \leq 1/\mathbb{E}[\xi(h)]\}}]\mathbb{E}[\xi(h)]^3 \\ &\qquad + 3\mathbb{E}[K(K-1)\mathbb{1}_{\{K \leq 1/\mathbb{E}[\xi(h)]\}}]\mathbb{E}[\xi(h)]\mathbb{E}[\xi(h)^2] + \mathbb{E}[K\mathbb{1}_{\{K \leq 1/\mathbb{E}[\xi(h)]\}}]\mathbb{E}[\xi(h)^3]\Big). \end{aligned} \tag{5.19}$$

By Lemma 3.4, for $\tau \in (3,5)$,

$$\mathbb{E}[K^3 \mathbb{1}_{\{K \leq 1/\mathbb{E}[\xi(h)]\}}] \leq C_{3,\tau}\mathbb{E}[\xi(h)]^{\tau-5}, \tag{5.20}$$

while, for $\tau = 5$,

$$\mathbb{E}[K^3 \mathbb{1}_{\{K \leq 1/\mathbb{E}[\xi(h)]\}}] \leq C_{3,5} \log\left(1/\mathbb{E}[\xi(h)]\right). \tag{5.21}$$

Similarly, by Lemma 3.4, for $\tau \in (3,4)$,

$$\mathbb{E}[K^2 \mathbb{1}_{\{K \leq 1/\mathbb{E}[\xi(h)]\}}] \leq C_{2,\tau}\mathbb{E}[\xi(h)]^{\tau-4}, \tag{5.22}$$

while, for $\tau = 4$,

$$\mathbb{E}[K^2 \mathbb{1}_{\{K \leq 1/\mathbb{E}[\xi(h)]\}}] \leq C_{2,4} \log\left(1/\mathbb{E}[\xi(h)]\right). \tag{5.23}$$

For the other terms we can replace the indicator function by 1 and use the upper bound on $\mathbb{E}[\xi(h)^2]$ of Lemma 5.1. For the second expectation in (5.18) we bound $\xi(x) \leq \beta$, so that this expectation is bounded from above by $\beta^3 C_{0,\tau}\mathbb{E}[\xi(h)]^{\tau-2}$. Combining these bounds and using $\mathbb{E}[\xi(h)]^3 \leq \log\left(1/\mathbb{E}[\xi(h)]\right)\mathbb{E}[\xi(h)]^3$ for $\tau = 5$, $\mathbb{E}[\xi(h)]^3 \leq \mathbb{E}[\xi(h)]^{\tau-2}$ for $\tau \in (4,5)$, $\mathbb{E}[\xi(h)]^3 \log\left(1/\mathbb{E}[\xi(h)]\right)^2 \leq \mathbb{E}[\xi(h)]^2$ for $\tau = 4$ and $\mathbb{E}[\xi(h)]^{2\tau-5} \leq \mathbb{E}[\xi(h)]^{\tau-2}$ for $\tau \in (3,4)$ gives the desired result. □

5.2 Bounds on first moment of $\xi(h)$

Proposition 5.3 (Upper bound on first moment of $\xi(h)$)**.** *Let $\beta \geq \beta_c$ and $B > 0$. Then, there exists a $C_1 > 0$ such that*

$$\mathbb{E}[\xi(h)] \leq \theta B + \theta\nu\mathbb{E}[\xi(h)] - C_1\mathbb{E}[\xi(h)]^{\boldsymbol{\delta}}, \tag{5.24}$$

where $\boldsymbol{\delta}$ takes the values as stated in Theorem 2.8. For $\tau = 5$,

$$\mathbb{E}[\xi(h)] \leq \theta B + \theta\nu\mathbb{E}[\xi(h)] - C_1\mathbb{E}[\xi(h)]^3 \log\left(1/\mathbb{E}[\xi(h)]\right). \tag{5.25}$$

Proof. We first use recursion (3.1) and rewrite it as

$$\mathbb{E}[\xi(h)] = \mathbb{E}\Big[\xi\Big(B + \sum_{i=1}^{K}\xi(h_i)\Big)\Big] = \theta B + \theta\nu\mathbb{E}[\xi(h)] + T_1 + T_2, \tag{5.26}$$

where

$$T_1 = \mathbb{E}\Big[\xi\Big(B + K\mathbb{E}[\xi(h)]\Big) - \theta\left(B + K\mathbb{E}[\xi(h)]\right)\Big], \tag{5.27}$$

and

$$T_2 = \mathbb{E}\Big[\xi\Big(B + \sum_{i=1}^{K}\xi(h_i)\Big) - \xi\left(B + K\mathbb{E}[\xi(h)]\right)\Big]. \tag{5.28}$$

Here, T_1 can be seen as the error of a first order Taylor series approximation of $\xi\left(B + K\mathbb{E}[\xi(h)]\right)$ around 0, whereas T_2 is the error made by replacing $\xi(h_i)$ by its expected value in the sum. By Lemma 3.2, $T_1 \leq 0$ and by concavity of $x \mapsto \xi(x)$ and Jensen's inequality $T_2 \leq 0$. We bound T_1 separately for the cases where $\mathbb{E}[K^3] < \infty$, $\tau \in (3,5)$ and $\tau = 5$. Since these bounds on T_1 are already of the correct order to prove the proposition it suffices to use $T_2 \leq 0$ and we do not bound T_2 more sharply.

Bound on T_1 when $\mathbb{E}[K^3] < \infty$. To bound T_1 for $\mathbb{E}[K^3] < \infty$ we use that $\xi''(0) = 0$, so that it follows from Taylor's theorem that, a.s.,

$$\xi\left(B + K\mathbb{E}[\xi(h)]\right) - \theta\left(B + K\mathbb{E}[\xi(h)]\right) = \frac{\xi'''(\zeta)}{6}\left(B + K\mathbb{E}[\xi(h)]\right)^3, \tag{5.29}$$

for some $\zeta \in (0, B + K\mathbb{E}[\xi(h)])$. Note that

$$\xi'''(\zeta) = -\frac{2\theta(1-\theta^2)(1-\tanh^2\zeta)}{(1-\theta^2\tanh^2\zeta)^3}\left(1 - 3(1-\theta^2)\tanh^2\zeta - \theta^2\tanh^4\zeta\right), \tag{5.30}$$

which we would like to bound from above by a negative constant. By Lemma 3.2, a.s.,

$$\xi\left(B + K\mathbb{E}[\xi(h)]\right) - \theta\left(B + K\mathbb{E}[\xi(h)]\right) \le 0, \tag{5.31}$$

so that we are allowed to assume that $B + K\mathbb{E}[\xi(h)]$ is sufficiently small. Indeed, from the above equation it follows that, for any constant a,

$$T_1 \le \mathbb{E}\left[\left(\xi\left(B + K\mathbb{E}[\xi(h)]\right) - \theta\left(B + K\mathbb{E}[\xi(h)]\right)\right)\mathbb{1}_{\{B+K\mathbb{E}[\xi(h)]\le a\}}\right]. \tag{5.32}$$

If, for convenience, we choose $a = \operatorname{atanh}\frac{1}{2}$, then

$$\xi'''(\zeta) \le -\frac{3}{8}\theta(1-\theta^2), \tag{5.33}$$

and hence,

$$\begin{aligned} T_1 &\le -\frac{1}{16}\theta(1-\theta^2)\mathbb{E}\left[\left(B + K\mathbb{E}[\xi(h)]\right)^3\mathbb{1}_{\{B+K\mathbb{E}[\xi(h)]\le \operatorname{atanh}\frac{1}{2}\}}\right] \\ &\le -\frac{1}{16}\theta(1-\theta^2)\mathbb{E}[K^3\mathbb{1}_{\{K\mathbb{E}[\xi(h)]\le \operatorname{atanh}\frac{1}{2}-B\}}]\mathbb{E}[\xi(h)]^3. \end{aligned} \tag{5.34}$$

Note that $B + K\mathbb{E}[\xi(h)]$ converges to zero for both limits of interest. Thus $\mathbb{E}[K^3\mathbb{1}_{\{K\mathbb{E}[\xi(h)]\le \operatorname{atanh}\frac{1}{2}-B\}}]$ goes to infinity in the case $\mathbb{E}[K^3] = \infty$ and hence this bound is not useful. We provide better bounds in the next paragraph for this case.

Bound on T_1 when $\tau \in (3, 5]$. For $\tau \in (3, 5]$, we make the expectation over K explicit:

$$T_1 = \sum_{k=0}^{\infty}\rho_k\left(\xi\left(B + k\mathbb{E}[\xi(h)]\right) - \theta\left(B + k\mathbb{E}[\xi(h)]\right)\right), \tag{5.35}$$

where it should be noted that all terms in this sum are negative because of Lemma 3.2. Define $f(k) = \xi\left(B + k\mathbb{E}[\xi(h)]\right) - \theta\left(B + k\mathbb{E}[\xi(h)]\right)$ and note that $f(k)$ is non-increasing. We use (3.16) and Lemma 3.3 to rewrite

$$\begin{aligned} T_1 = \sum_{k=0}^{\infty} f(k)\rho_k &= f(0) + \sum_{k\ge1}[f(k) - f(k-1)]\rho_{\ge k} \le f(0) + c_\rho\sum_{k\ge1}[f(k) - f(k-1)]k^{-(\tau-2)} \\ &= f(0) + c_\rho\sum_{k\ge1}[f(k) - f(k-1)]\sum_{\ell\ge k}(\ell^{-(\tau-2)} - (\ell+1)^{-(\tau-2)}) \end{aligned} \tag{5.36}$$

Then, we can again interchange the summation order as we did to obtain (3.16) to rewrite this as

$$\begin{aligned} T_1 &\leq f(0) + c_\rho \sum_{\ell\geq 1}\sum_{k=1}^{\ell}[f(k)-f(k-1)](\ell^{-(\tau-2)}-(\ell+1)^{-(\tau-2)}) \\ &= f(0)(1-c_\rho) + c_\rho \sum_{\ell\geq 1} f(\ell)(\ell^{-(\tau-2)}-(\ell+1)^{-(\tau-2)}). \end{aligned} \tag{5.37}$$

Using the convexity of $\ell^{-(\tau-2)}$ this can be bounded as

$$T_1 \leq f(0)(1-c_\rho) + (\tau-2)c_\rho \sum_{\ell\geq 1} f(\ell)(\ell+1)^{-(\tau-1)}. \tag{5.38}$$

Since we can assume that $c_\rho \leq 1$, $f(0)(1-c_\rho)\leq 0$. Hence,

$$\begin{aligned} T_1 &\leq (\tau-2)c_\rho\left(\mathbb{E}[\xi(h)]\right)^{\tau-1}\sum_{k=0}^{\infty}((k+1)\mathbb{E}[\xi(h)])^{-(\tau-1)}\left(\xi\left(B+k\mathbb{E}[\xi(h)]\right)-\theta\left(B+k\mathbb{E}[\xi(h)]\right)\right) \\ &\leq (\tau-2)c_\rho\left(\mathbb{E}[\xi(h)]\right)^{\tau-1}\sum_{k=a/\mathbb{E}[\xi(h)]}^{b/\mathbb{E}[\xi(h)]}(k\mathbb{E}[\xi(h)])^{-(\tau-1)}\left(\xi\left(B+k\mathbb{E}[\xi(h)]\right)-\theta\left(B+k\mathbb{E}[\xi(h)]\right)\right), \end{aligned} \tag{5.39}$$

where we choose a and b such that $0<a<b<\infty$. We use dominated convergence on the above sum. The summands are uniformly bounded, and $\mathbb{E}[\xi(h)]\to 0$ for both limits of interest. Further, when $k\mathbb{E}[\xi(h)]=y$, the summand converges pointwise to $y^{-(\tau-1)}\left(\xi\left(B+y\right)-\theta\left(B+y\right)\right)$. Hence, we can write the sum above as

$$\mathbb{E}[\xi(h)]^{-1}\left(\int_a^b y^{-(\tau-1)}\left(\xi\left(B+y\right)-\theta\left(B+y\right)\right)\mathrm{d}y + o(1)\right), \tag{5.40}$$

where $o(1)$ is a function tending to zero for both limits of interest [22, 216 A]. The integrand is uniformly bounded for $y\in[a,b]$ and hence we can bound the integral from above by a (negative) constant $-I$ for B sufficiently small and β sufficiently close to β_c. Hence,

$$\mathbb{E}[\xi(h)] \leq \theta B + \theta\nu\mathbb{E}[\xi(h)] - (\tau-1)c_\rho I\mathbb{E}[\xi(h)]^{\tau-2}. \tag{5.41}$$

Logarithmic corrections in the bound for $\tau=5$. We complete the proof by identifying the logarithmic correction for $\tau=5$. Since the random variable in the expectation in T_1 is non-positive, we can bound

$$T_1 \leq \mathbb{E}\left[\xi\left(B+K\mathbb{E}[\xi(h)]\right)-\theta\left(B+K\mathbb{E}[\xi(h)]\right)\mathbb{1}_{\{K\leq\varepsilon/\mathbb{E}[\xi(h)]\}}\right]. \tag{5.42}$$

Taylor expansion $h\mapsto\xi(h)$ to third order, using that $\xi(0)=\xi''(0)=0$, while the linear term cancels, leads to

$$T_1 \leq \mathbb{E}\left[\frac{\xi'''(\zeta)}{6}\left(B+K\mathbb{E}[\xi(h)]\right)^3\mathbb{1}_{\{K\leq\varepsilon/\mathbb{E}[\xi(h)]\}}\right], \tag{5.43}$$

for some $\zeta \in (0, K\mathbb{E}[\xi(h)])$. On the event that $K \leq \varepsilon/\mathbb{E}[\xi(h)]$, we thus have that $\zeta \in (0, \varepsilon)$, and $\xi'''(\zeta) \leq -c_\varepsilon \equiv \sup_{x\in(0,\varepsilon)} \xi'''(x) < 0$ when ε is sufficiently small. Thus,

$$\begin{aligned} T_1 &\leq -\frac{c_\varepsilon}{6}\mathbb{E}\left[(B + K\mathbb{E}[\xi(h)])^3 \mathbb{1}_{\{K\leq\varepsilon/\mathbb{E}[\xi(h)]\}}\right] \\ &\leq -\frac{c_\varepsilon}{6}\mathbb{E}[\xi(h)]^3\mathbb{E}\left[K(K-1)(K-2)\mathbb{1}_{\{K\leq\varepsilon/\mathbb{E}[\xi(h)]\}}\right]. \end{aligned} \tag{5.44}$$

When $\tau = 5$, by Lemma 3.4, $\mathbb{E}\left[K(K-1)(K-2)\mathbb{1}_{\{K\leq\ell\}}\right] \geq c_{3,5}\log\ell$, which completes the proof. $\square$

Proposition 5.4 (Lower bound on first moment of $\xi(h)$)**.** *Let $\beta \geq \beta_c$ and $B > 0$. Then, there exists a constant $C_2 > 0$ such that*

$$\mathbb{E}[\xi(h)] \geq \theta B + \theta\nu\mathbb{E}[\xi(h)] - c_1\mathbb{E}[\xi(h)]^{\boldsymbol{\delta}} - Be_1, \tag{5.45}$$

where

$$\boldsymbol{\delta} = \begin{cases} 3 & \text{when } \mathbb{E}[K^3] < \infty, \\ \tau - 2 & \text{when } \tau \in (3,5). \end{cases} \tag{5.46}$$

For $\tau = 5$,

$$\mathbb{E}[\xi(h)] \geq \theta B + \theta\nu\mathbb{E}[\xi(h)] - c_1\mathbb{E}[\xi(h)]^3\log(1/\mathbb{E}[\xi(h)]) - Be_1. \tag{5.47}$$

Proof. We again use the split in (5.26) and we bound T_1 and T_2.

The lower bound on T_1. For $\mathbb{E}[K^3] < \infty$, we use the lower bound of Lemma 3.2 to get

$$T_1 \geq -\frac{\theta}{3(1-\theta^2)}\mathbb{E}\left[(B + K\mathbb{E}[\xi(h)])^3\right]. \tag{5.48}$$

By expanding, this can be rewritten as

$$T_1 \geq -\frac{\theta}{3(1-\theta^2)}\mathbb{E}[K^3]\mathbb{E}[\xi(h)]^3 - Be_4. \tag{5.49}$$

For $\tau \in (3,5]$, we first split T_1 in a small K and a large K part. For this, write

$$t_1(k) = \xi\left(B + k\mathbb{E}[\xi(h)]\right) - \theta\left(B + k\mathbb{E}[\xi(h)]\right). \tag{5.50}$$

Then,

$$T_1 = \mathbb{E}[t_1(K)] = \mathbb{E}\left[t_1(K)\mathbb{1}_{\{K\leq\varepsilon/\mathbb{E}[\xi(h)]\}}\right] + \mathbb{E}\left[t_1(K)\mathbb{1}_{\{K>\varepsilon/\mathbb{E}[\xi(h)]\}}\right]. \tag{5.51}$$

To bound the first term, we again use (5.48):

$$\mathbb{E}\left[t_1(K)\mathbb{1}_{\{K\leq\varepsilon/\mathbb{E}[\xi(h)]\}}\right] \geq -\frac{\theta}{3(1-\theta^2)}\mathbb{E}\left[(B + K\mathbb{E}[\xi(h)])^3\mathbb{1}_{\{K\leq\varepsilon/\mathbb{E}[\xi(h)]\}}\right]. \tag{5.52}$$

It is easy to see that the terms $B^3\mathbb{E}\left[\mathbb{1}_{\{K>\varepsilon/\mathbb{E}[\xi(h)]\}}\right]$ and $3B^2\mathbb{E}[\xi(h)]\mathbb{E}\left[K\mathbb{1}_{\{K\leq\varepsilon/\mathbb{E}[\xi(h)]\}}\right]$ that we get by expanding the above are of the form Be. To bound the other two terms, we use Lemma 3.4 to obtain, for $\varepsilon \leq 1$,

$$3B\mathbb{E}[\xi(h)]^2\mathbb{E}\left[K^2\mathbb{1}_{\{K\leq\varepsilon/\mathbb{E}[\xi(h)]\}}\right] \leq \begin{cases} 3B\mathbb{E}[\xi(h)]^2\mathbb{E}\left[K^2\right] & \text{when } \tau\in(4,5], \\ 3BC_{2,4}\mathbb{E}[\xi(h)]^2\log(1/\mathbb{E}[\xi(h)]) & \text{when } \tau=4, \\ 3BC_{2,\tau}\mathbb{E}[\xi(h)]^{\tau-2} & \text{when } \tau\in(3,4), \end{cases} \tag{5.53}$$

which are all of the form Be, and

$$\mathbb{E}\left[K^3\mathbb{1}_{\{K\leq\varepsilon/\mathbb{E}[\xi(h)]\}}\right]\mathbb{E}[\xi(h)]^3 \leq \begin{cases} C_{3,5}\mathbb{E}[\xi(h)]^3\log(1/\mathbb{E}[\xi(h)]) & \text{when } \tau=5, \\ C_{3,\tau}\mathbb{E}[\xi(h)]^{\tau-2} & \text{when } \tau\in(3,5). \end{cases} \tag{5.54}$$

To bound T_1 for large K, we observe that

$$\mathbb{E}\left[t_1(K)\mathbb{1}_{\{K>\varepsilon/\mathbb{E}[\xi(h)]\}}\right] \geq -\theta B\mathbb{E}[\mathbb{1}_{\{K>\varepsilon/\mathbb{E}[\xi(h)]\}}] - \theta\mathbb{E}[\xi(h)]\mathbb{E}[K\mathbb{1}_{\{K>\varepsilon/\mathbb{E}[\xi(h)]\}}]. \tag{5.55}$$

Applying Lemma 3.4 now gives, for $\tau\in(3,5]$

$$\begin{aligned}\mathbb{E}\left[t_1(K)\mathbb{1}_{\{K>\varepsilon/\mathbb{E}[\xi(h)]\}}\right] &\geq -\theta BC_{0,\tau}\mathbb{E}[\xi(h)]^{\tau-2} - \theta C_{1,\tau}\mathbb{E}[\xi(h)]^{\tau-2} \\ &= -C_4\mathbb{E}[\xi(h)]^{\tau-2} - Be_4. \end{aligned} \tag{5.56}$$

The lower bound on T_2. To bound T_2, we split in a small and a large K contribution:

$$T_2 = \mathbb{E}[t_2(K)\mathbb{1}_{\{K\leq\varepsilon/\mathbb{E}[\xi(h)]\}}] + \mathbb{E}[t_2(K)\mathbb{1}_{\{K>\varepsilon/\mathbb{E}[\xi(h)]\}}] \equiv T_2^{\leq} + T_2^{>}, \tag{5.57}$$

where

$$t_2(k) = \xi\left(B + \sum_{i=1}^{k}\xi(h_i)\right) - \xi\left(B + k\mathbb{E}[\xi(h)]\right). \tag{5.58}$$

To bound $T_2^{>}$, we note that

$$t_2(k) \geq -\beta, \tag{5.59}$$

so that

$$T_2^{>} \geq -\beta\mathbb{E}[\mathbb{1}_{\{K>\varepsilon/\mathbb{E}[\xi(h)]\}}] \geq -C_5\mathbb{E}[\xi(h)]^{(\tau-2)\wedge 3}, \tag{5.60}$$

where we have used Lemma 3.4 in the last inequality and the Markov inequality when $\mathbb{E}[K^3]<\infty$.

To bound $T_2^{\leq}$, we start from

$$T_2^{\leq} = \mathbb{E}\left[\frac{\xi''(\zeta)}{2}\left(\sum_{i=1}^{K}\xi(h_i) - K\mathbb{E}[\xi(h)]\right)^2\mathbb{1}_{\{K\leq\varepsilon/\mathbb{E}[\xi(h)]\}}\right], \tag{5.61}$$

for some ζ in between $B+\sum_{i=1}^{K}\xi(h_i)$ and $B+K\mathbb{E}[\xi(h)]$. We use that

$$\xi''(\zeta) \geq -\frac{2\theta}{1-\theta^2}\Big(B + \sum_{i=1}^{K}\xi(h_i) + K\mathbb{E}[\xi(h)]\Big). \tag{5.62}$$

to obtain

$$
\begin{aligned}
T_2^{\leq} &\geq -\frac{\theta}{1-\theta^2}\mathbb{E}\Big[\Big(B+\sum_{i=1}^{K}\xi(h_i)+K\mathbb{E}[\xi(h)]\Big)\Big(\sum_{i=1}^{K}\xi(h_i)-K\mathbb{E}[\xi(h)]\Big)^2\mathbb{1}_{\{K\leq\varepsilon/\mathbb{E}[\xi(h)]\}}\Big] \\
&\geq -\frac{\theta}{1-\theta^2}\Big(B\nu\mathbb{E}\left[(\xi(h)-\mathbb{E}[\xi(h)])^2\right]+\nu\mathbb{E}\left[(\xi(h)+\mathbb{E}[\xi(h)])\,(\xi(h)-\mathbb{E}[\xi(h)])^2\right] \\
&\qquad +2\mathbb{E}[K^2\mathbb{1}_{\{K\leq\varepsilon/\mathbb{E}[\xi(h)]\}}]\mathbb{E}[\xi(h)]\mathbb{E}\left[(\xi(h)-\mathbb{E}[\xi(h)])^2\right]\Big) \\
&\geq -\frac{\theta}{1-\theta^2}\Big(B\nu\mathbb{E}[\xi(h)^2]+\left(2\mathbb{E}[K^2\mathbb{1}_{\{K\leq\varepsilon/\mathbb{E}[\xi(h)]\}}]+\nu\right)\mathbb{E}[\xi(h)]\mathbb{E}[\xi(h)^2]+\nu\mathbb{E}[\xi(h)^3]\Big).
\end{aligned}
\tag{5.63}
$$

Using the bounds of Lemmas 3.4, 5.1 and 5.2 we get,

$$
T_2^{\leq} \geq \begin{cases}
-\frac{\theta}{1-\theta^2}\left(\left(2\mathbb{E}[K^2]+\nu\right)C_2+C_3\nu\right)\mathbb{E}[\xi(h)]^3-Be_5 & \text{when } \mathbb{E}[K^3]<\infty, \\
-\frac{\theta}{1-\theta^2}\left(\left(2\mathbb{E}[K^2]+\nu\right)C_2+C_3\nu\right)\mathbb{E}[\xi(h)]^3\log(1/\mathbb{E}[\xi(h)])-Be_5 & \text{when } \tau=5, \\
-\frac{\theta}{1-\theta^2}\left(C_{2,\tau}'+C_3\nu\right)\mathbb{E}[\xi(h)]^{\tau-2}-Be_5 & \text{when } \tau\in(3,5),
\end{cases}
\tag{5.64}
$$

where $C_{2,\tau}' = \left(2\mathbb{E}[K^2]+\nu\right)C_2$ for $\tau\in(4,5)$ and $C_{2,\tau}'=(2C_{2,\tau}+\nu)C_2$ for $\tau\in(3,4]$. Here, we have also used that (a) $\mathbb{E}[\xi(h)]^3\leq\mathbb{E}[\xi(h)]^3\log(1/\mathbb{E}[\xi(h)])$ for $\tau=5$; (b) $\mathbb{E}[\xi(h)]^3\leq\mathbb{E}[\xi(h)]^{\tau-2}$ for $\tau\in(4,5]$; (c) $\mathbb{E}[\xi(h)]^3\log\left(1/\mathbb{E}[\xi(h)]\right)^2\leq\mathbb{E}[\xi(h)]^2$ and $\mathbb{E}[\xi(h)]^3\log\left(1/\mathbb{E}[\xi(h)]\right)\leq\mathbb{E}[\xi(h)]^2$ for $\tau=4$; and (d) $\mathbb{E}[\xi(h)]^{\tau-1}\leq\mathbb{E}[\xi(h)]^{\tau-2}$ and $\mathbb{E}[\xi(h)]^{2\tau-5}\leq\mathbb{E}[\xi(h)]^{\tau-2}$ for $\tau\in(3,4)$. Combining the bounds on T_1 and T_2 gives the desired lower bound on $\mathbb{E}[\xi(h)]$. □

5.3 Critical exponents $\boldsymbol{\beta}$ and $\boldsymbol{\delta}$

It remains to show that the bounds on $\mathbb{E}[\xi(h)]$ give us the desired result:

Theorem 5.5 (Values of $\boldsymbol{\beta}$ and $\boldsymbol{\delta}$)**.** *The critical exponents $\boldsymbol{\beta}$ and $\boldsymbol{\delta}$ equal the values as stated in Theorem 2.8 when $\mathbb{E}[K^3]<\infty$ and $\tau\in(3,5)$. Furthermore, for $\tau=5$, (2.20) holds.*

Proof. We prove the upper and the lower bounds separately, starting with the upper bound.

The upper bounds on the magnetization. We start by bounding the magnetization from above:

$$
M(\beta,B)=\mathbb{E}\left[\tanh\left(B+\sum_{i=1}^{D}\xi(h_i)\right)\right]\leq B+\mathbb{E}[D]\mathbb{E}[\xi(h)]. \tag{5.65}
$$

We first perform the analysis for $\boldsymbol{\beta}$. Taking the limit $B\searrow 0$ in (5.24) in Proposition 5.3 yields

$$
\mathbb{E}[\xi(h_0)]\leq\theta\nu\mathbb{E}[\xi(h_0)]-C_1\mathbb{E}[\xi(h_0)]^{\boldsymbol{\delta}}, \tag{5.66}
$$

where $h_0=h(\beta,0^+)$. For $\beta>\beta_c$, by definition, $\mathbb{E}[\xi(h_0)]>0$ and thus we can divide through by $\mathbb{E}[\xi(h_0)]$ to obtain

$$
\mathbb{E}[\xi(h_0)]^{\boldsymbol{\delta}-1}\leq\frac{\theta\nu-1}{C_1}. \tag{5.67}
$$

By Taylor's theorem,

$$\theta\nu - 1 \leq \nu(1-\theta_c^2)(\beta - \beta_c). \tag{5.68}$$

Hence,

$$\mathbb{E}[\xi(h_0)] \leq \left(\frac{\nu(1-\theta_c^2)}{C_1}\right)^{1/(\boldsymbol{\delta}-1)} (\beta-\beta_c)^{1/(\boldsymbol{\delta}-1)}. \tag{5.69}$$

Using that $\boldsymbol{\beta} = 1/(\boldsymbol{\delta}-1)$,

$$M(\beta, 0^+) \leq \mathbb{E}[D] \left(\frac{\nu(1-\theta_c^2)}{C_1}\right)^{\boldsymbol{\beta}} (\beta-\beta_c)^{\boldsymbol{\beta}}, \tag{5.70}$$

from which it easily follows that

$$\limsup_{\beta \searrow \beta_c} \frac{M(\beta, 0^+)}{(\beta-\beta_c)^{\boldsymbol{\beta}}} < \infty. \tag{5.71}$$

We complete the analysis for $\boldsymbol{\beta}$ by analyzing $\tau = 5$. Since (5.24) also applies to $\tau = 5$, (5.71) holds as well. We now improve upon this using (5.25) in Proposition 5.3, which yields in a similar way as above that

$$\mathbb{E}[\xi(h_0)]^2 \leq \frac{\theta\nu - 1}{C_1 \log(1/\mathbb{E}[\xi(h_0)])}. \tag{5.72}$$

Since $x \mapsto 1/\log(1/x)$ is increasing on $(0,1)$ and $\mathbb{E}[\xi(h_0)] \leq C(\beta-\beta_c)^{1/2}$ for some $C > 0$, we immediately obtain that

$$\mathbb{E}[\xi(h_0)]^2 \leq \frac{\theta\nu - 1}{C_1 \log(1/\mathbb{E}[\xi(h_0)])} \leq \frac{\theta\nu - 1}{C_1 \log(1/[C(\beta-\beta_c)^{1/2}])}. \tag{5.73}$$

Taking the limit of $\beta \searrow \beta_c$ as above then completes the proof.

We continue with the analysis for $\boldsymbol{\delta}$. Setting $\beta = \beta_c$ in (5.24) and rewriting gives

$$\mathbb{E}[\xi(h_c)] \leq \left(\frac{\theta_c}{C_1}\right)^{1/\boldsymbol{\delta}} B^{1/\boldsymbol{\delta}}, \tag{5.74}$$

with $h_c = h(\beta_c, B)$. Hence,

$$M(\beta_c, B) \leq B + \mathbb{E}[D] \left(\frac{\theta_c}{C_1}\right)^{1/\boldsymbol{\delta}} B^{1/\boldsymbol{\delta}}, \tag{5.75}$$

so that, using $1/\boldsymbol{\delta} < 1$,

$$\limsup_{B \searrow 0} \frac{M(\beta_c, B)}{B^{1/\boldsymbol{\delta}}} < \infty. \tag{5.76}$$

The analysis for $\boldsymbol{\delta}$ for $\tau = 5$ can be performed in an identical way as for $\boldsymbol{\beta}$.

The lower bounds on the magnetization. For the lower bound on the magnetization we use that

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\tanh x = -2\tanh x(1-\tanh^2 x) \geq -2, \tag{5.77}$$

so that

$$\tanh x \geq x - x^2. \tag{5.78}$$

Hence,

$$\begin{aligned} M(\beta,B) &\geq B + \mathbb{E}[D]\mathbb{E}[\xi(h)] - \mathbb{E}\Big[\Big(B + \sum_{i=1}^{D}\xi(h_i)\Big)^2\Big] \\ &\geq B + \mathbb{E}[D]\mathbb{E}[\xi(h)] - Be_6 - \mathbb{E}[D(D-1)]\mathbb{E}[\xi(h)]^2 - \mathbb{E}[D]C_2\mathbb{E}[\xi(h)]^{2\wedge(\tau-2)} \\ &= B + (\mathbb{E}[D]-e_7)\mathbb{E}[\xi(h)] - Be_6, \end{aligned} \tag{5.79}$$

with $a \wedge b$ denoting the minimum of a and b, where e_7 satisfies (5.2) both also $\liminf_{\beta\searrow\beta_c} e_7(\beta,0^+) = 0$, because $\mathbb{E}[\xi(h)]$ converges to zero for both limits of interest.

We again first perform the analysis for $\boldsymbol{\beta}$ and $\tau \neq 5$. We get from (5.45) in Proposition 5.4 that

$$\mathbb{E}[\xi(h_0)] \geq \left(\frac{\theta\nu - 1}{c_1}\right)^{1/(\boldsymbol{\delta}-1)} \geq \left(\frac{\nu(1-\theta^2)}{c_1}\right)^{\boldsymbol{\beta}} (\beta-\beta_c)^{\boldsymbol{\beta}}, \tag{5.80}$$

where the last inequality holds because, by Taylor's theorem,

$$\theta\nu - 1 \geq \nu(1-\theta^2)(\beta-\beta_c). \tag{5.81}$$

Hence,

$$M(\beta,0^+) \geq (\mathbb{E}[D]-e_7)\left(\frac{\nu(1-\theta^2)}{c_1}\right)^{\boldsymbol{\beta}} (\beta-\beta_c)^{\boldsymbol{\beta}}, \tag{5.82}$$

so that

$$\liminf_{\beta\searrow\beta_c} \frac{M(\beta,0^+)}{(\beta-\beta_c)^{\boldsymbol{\beta}}} \geq \mathbb{E}[D]\left(\frac{\nu(1-\theta^2)}{c_1}\right)^{\boldsymbol{\beta}} > 0. \tag{5.83}$$

For $\tau = 5$, we note that (5.47) as well as the fact that $\log 1/x \leq A_\varepsilon x^{-\varepsilon}$ for all $x \in (0,1)$ and some $A_\varepsilon > 0$, yields that

$$\mathbb{E}[\xi(h_0)] \geq \left(\frac{\theta\nu - 1}{A_\varepsilon c_1}\right)^{1/(2+\varepsilon)} \geq \left(\frac{\nu(1-\theta^2)}{A_\varepsilon c_1}\right)^{1/(2+\varepsilon)} (\beta-\beta_c)^{1/(2+\varepsilon)}. \tag{5.84}$$

Then again using (5.47) yields, for some constant $c > 0$,

$$\mathbb{E}[\xi(h_0)] \geq \left(\frac{\theta\nu - 1}{c_1 \log(1/\mathbb{E}[\xi(h_0)])}\right)^{1/2} \geq c\Big(\frac{\beta-\beta_c}{\log(1/(\beta-\beta_c))}\Big)^{1/2}, \tag{5.85}$$

once more since $x \mapsto 1/(\log(1/x))$ is increasing.

We continue with the analysis for $\boldsymbol{\delta}$. Again, setting $\beta = \beta_c$ in (5.45), we get

$$\mathbb{E}[\xi(h_c)] \geq \left(\frac{\theta_c - e_1}{c_1}\right)^{1/\boldsymbol{\delta}} B^{1/\boldsymbol{\delta}}, \tag{5.86}$$

from which it follows that

$$M(\beta_c, B) \geq (\mathbb{E}[D] - e_7)\left(\frac{\theta_c - e_1}{c_1}\right)^{1/\boldsymbol{\delta}} B^{1/\boldsymbol{\delta}} - Be_6, \tag{5.87}$$

and hence,

$$\liminf_{B \searrow 0} \frac{M(\beta_c, B)}{B^{1/\boldsymbol{\delta}}} \geq \mathbb{E}[D]\left(\frac{\theta_c}{c_1}\right)^{1/\boldsymbol{\delta}} > 0, \tag{5.88}$$

as required. The extension to $\tau = 5$ can be dealt with in an identical way as in (5.84)–(5.85). This proves the theorem. $\square$

We can now also derive the joint scaling of the magnetization as $(\beta, B) \searrow (\beta_c, 0)$:

Proof of Corollary 2.9. We start with the lower bound for $\tau \neq 5$. From the GKS inequality it follows that, for $\beta > \beta_c$ and $B > 0$, $M(\beta, B) \geq \frac{1}{2}(M(\beta_c, B) + M(\beta, 0^+))$. Combined with the lower bounds in (5.82) and (5.87) this gives

$$M(\beta, B) \geq c_6(\beta - \beta_c)^{\boldsymbol{\beta}} + c_7 B^{1/\boldsymbol{\delta}}, \tag{5.89}$$

which gives the desired lower bound.

For the upper bound for $\tau \neq 5$ we rewrite (5.24) as

$$C_1 \mathbb{E}[\xi(h)]^{\boldsymbol{\delta}} \leq B + (\theta\nu - 1)\mathbb{E}[\xi(h)]. \tag{5.90}$$

Dividing both sides by $\mathbb{E}[\xi(h)]$, which is allowed for $\beta > \beta_c$ and $B > 0$, and using (5.68) gives

$$C_1 \mathbb{E}[\xi(h)]^{\boldsymbol{\delta}-1} \leq \frac{B}{\mathbb{E}[\xi(h)]} + \nu(1 - \theta_c^2)(\beta - \beta_c). \tag{5.91}$$

Since, by the GKS inequality and (5.86),

$$\mathbb{E}[\xi(h)] \geq \mathbb{E}[\xi(h_c)] \geq c_8 B^{1/\boldsymbol{\delta}}, \tag{5.92}$$

we thus have that

$$\mathbb{E}[\xi(h)] \leq \left(\frac{1}{c_8 C_1} B^{(\boldsymbol{\delta}-1)/\boldsymbol{\delta}} + \frac{\nu}{C_1}(1 - \theta_c^2)(\beta - \beta_c)\right)^{1/(\boldsymbol{\delta}-1)} \leq C_6 B^{1/\boldsymbol{\delta}} + C_7(\beta - \beta_c)^{\boldsymbol{\beta}}, \tag{5.93}$$

where we used $(x + y)^{1/(\boldsymbol{\delta}-1)} \leq (2(x \vee y))^{1/(\boldsymbol{\delta}-1)} \leq 2^{1/(\boldsymbol{\delta}-1)} x^{1/(\boldsymbol{\delta}-1)} + 2^{1/(\boldsymbol{\delta}-1)} y^{1/(\boldsymbol{\delta}-1)}$, with $x \vee y$ denoting the maximimum of x and y, in the second inequality. The result for $\tau \neq 5$ now follows from (5.65). The proof for $\tau = 5$ is similar. $\square$

6 Critical exponents: Susceptibility

In this section, we study the susceptibility. In Section 6.1 we identify $\boldsymbol{\gamma}$, in Section 6.2 we prove a lower bound on $\boldsymbol{\gamma}'$ and add a heuristic why this is the correct value.

6.1 The critical exponent $\boldsymbol{\gamma}$

For the susceptibility in the *subcritical* phase, i.e., in the high-temperature region $\beta < \beta_c$, we can not only identify the critical exponent $\boldsymbol{\gamma}$, but we can also identify the constant:

Theorem 6.1 (Critical exponent $\boldsymbol{\gamma}$)**.** *For $\mathbb{E}[K] < \infty$ and $\beta < \beta_c$,*

$$\chi(\beta, 0^+) = 1 + \frac{\mathbb{E}[D]\theta}{1 - \nu\theta}. \tag{6.1}$$

In particular,

$$\lim_{\beta \nearrow \beta_c} \chi(\beta, 0^+)(\beta_c - \beta) = \frac{\mathbb{E}[D]\theta_c^2}{1 - \theta_c^2}, \tag{6.2}$$

and hence

$$\boldsymbol{\gamma} = 1. \tag{6.3}$$

Proof. The proof is divided into three steps. We first reduce the suceptibility on the random graph to the one on the random Bethe tree. Secondly, we rewrite the susceptibility on the tree using transfer matrix techniques. Finally, we use this rewrite (which applies to *all* β and $B > 0$) to prove that $\boldsymbol{\gamma} = 1$.

Reduction to the random tree. Let ϕ denote a vertex selected uniformly at random from $[n]$ and let $\mathbb{E}_\phi$ denote its expectation. Then we can write the susceptibility as

$$\chi_n \equiv \frac{1}{n} \sum_{i,j=1}^{n} \Big(\langle \sigma_i \sigma_j \rangle_{\mu_n} - \langle \sigma_i \rangle_{\mu_n} \langle \sigma_j \rangle_{\mu_n} \Big) = \mathbb{E}_\phi \left[\sum_{j=1}^{n} \Big(\langle \sigma_\phi \sigma_j \rangle_{\mu_n} - \langle \sigma_\phi \rangle_{\mu_n} \langle \sigma_j \rangle_{\mu_n} \Big) \right]. \tag{6.4}$$

Note that

$$\langle \sigma_i \sigma_j \rangle_{\mu_n} - \langle \sigma_i \rangle_{\mu_n} \langle \sigma_j \rangle_{\mu_n} = \frac{\partial \langle \sigma_i \rangle_{\mu_n}}{\partial B_j}, \tag{6.5}$$

which is, by the GHS inequality [20], decreasing in external fields at all other vertices $k \in [n]$. Denote by $\langle \cdot \rangle^{t,+/f}$ the Ising model with +/free boundary conditions, respectively, at all vertices at graph distance t from ϕ. Then, for all $t \geq 1$,

$$\chi_n \geq \mathbb{E}_\phi \left[\sum_{j=1}^{n} \Big(\langle \sigma_\phi \sigma_j \rangle_{\mu_n}^{t,+} - \langle \sigma_\phi \rangle_{\mu_n}^{t,+} \langle \sigma_j \rangle_{\mu_n}^{t,+} \Big) \right]. \tag{6.6}$$

By introducing boundary conditions, only vertices in the ball $B_\phi(t)$ contribute to the sum. Hence, by taking the limit $n \to \infty$ and using that the graph is locally tree-like,

$$\chi \geq \mathbb{E} \left[\sum_{j \in T_t} \Big(\langle \sigma_\phi \sigma_j \rangle^{t,+} - \langle \sigma_\phi \rangle^{t,+} \langle \sigma_j \rangle^{t,+} \Big) \right], \tag{6.7}$$

where the expectation now is over the random tree $T_t \sim \mathcal{T}(D, K, t)$ with root ϕ.

For an upper bound on χ_n we use a trick similar to one used in the proof of [11, Corollary 4.5]: Let $B'_j = B$ if $j \in B_t(\phi)$ and $B'_j = B + H$ if $j \notin B_t(\phi)$ for some $H > -B$. Denote by $\langle \cdot \rangle_H$ the associated Ising expectation. Then, because of (6.5),

$$\mathbb{E}_\phi \left[\sum_{j \notin B_t(\phi)} \Big(\langle \sigma_\phi \sigma_j \rangle - \langle \sigma_\phi \rangle \langle \sigma_j \rangle \Big) \right] = \mathbb{E}_\phi \left[\frac{\partial}{\partial H} \langle \sigma_\phi \rangle_H \Big|_{H=0} \right], \tag{6.8}$$

By the GHS inequality, $\langle \sigma_\phi \rangle_H$ is a concave function of H and hence,

$$\mathbb{E}_\phi \left[\frac{\partial}{\partial H} \langle \sigma_\phi \rangle_H \Big|_{H=0} \right] \le \mathbb{E}_\phi \left[\frac{2}{B} \left(\langle \sigma_\phi \rangle_{H=0} - \langle \sigma_\phi \rangle_{H=-B/2} \right) \right]. \tag{6.9}$$

Using the GKS inequality this can be bounded from above by

$$\mathbb{E}_\phi \left[\frac{2}{B} \Big(\langle \sigma_\phi \rangle^{t,+}_{H=0} - \langle \sigma_\phi \rangle^{t,f}_{H=-B/2} \Big) \right] = \mathbb{E}_\phi \left[\frac{2}{B} \left(\langle \sigma_\phi \rangle^{t,+} - \langle \sigma_\phi \rangle^{t,f} \right) \right], \tag{6.10}$$

where the equality holds because the terms depend only on the system in the ball $B_t(\phi)$ and hence not on H. By letting $n \to \infty$, by the locally tree-likeness, this is equal to

$$\frac{2}{B} \mathbb{E} \left[\left(\langle \sigma_\phi \rangle^{t,+} - \langle \sigma_\phi \rangle^{t,f} \right) \right], \tag{6.11}$$

where the expectation and the Ising model now is over the random tree $T_t \sim \mathcal{T}(D, K, t)$ with root ϕ. From [15, Lemma 3.1] we know that this expectation can be bounded from above by M/t for some constant $M = M(\beta, B) < \infty$. Hence, if $t \to \infty$,

$$\lim_{t\to\infty} \mathbb{E} \left[\sum_{j \in T_t} \Big(\langle \sigma_\phi \sigma_j \rangle^{t,+} - \langle \sigma_\phi \rangle^{t,+} \langle \sigma_j \rangle^{t,+} \Big) \right] \le \chi \le \lim_{t\to\infty} \mathbb{E} \left[\sum_{j \in T_t} \Big(\langle \sigma_\phi \sigma_j \rangle^{t,f} - \langle \sigma_\phi \rangle^{t,f} \langle \sigma_j \rangle^{t,f} \Big) \right]. \tag{6.12}$$

Rewrite of the susceptibility on trees. It remains to study the susceptibility on trees. For this, condition on the tree T_∞. Then, for some vertex j at height $\ell \le t$ in the tree, denote the vertices on the unique path from ϕ to j by $\phi = v_0, v_1, \ldots, v_\ell = j$ and let, for $0 \le i \le \ell$, $S_{\le i} = (\sigma_{v_0}, \ldots, \sigma_{v_i})$. We first compute the expected value of a spin σ_{v_i} on this path, conditioned on the spin values $S_{\le i-1}$. Note that under this conditioning the expected spin value only depends on the spin value $\sigma_{v_{i-1}}$ and the effective field $h_{v_i} = h_{v_i}^{t,+/f}$ obtained by pruning the tree at vertex v_i, i.e., by removing all edges at vertex v_i going away from the root and replacing the external magnetic field at vertex v_i by h_{v_i} which can be exactly computed using [11, Lemma 4.1]. Hence,

$$\langle \sigma_{v_i} | S_{\le i-1} \rangle^{t,+/f} = \frac{e^{\beta \sigma_{v_{i-1}} + h_{v_i}} - e^{-\beta \sigma_{v_{i-1}} - h_{v_i}}}{e^{\beta \sigma_{v_{i-1}} + h_{v_i}} + e^{-\beta \sigma_{v_{i-1}} - h_{v_i}}}. \tag{6.13}$$

We can write the indicators $\mathbb{1}_{\{\sigma_{v_{i-1}} = \pm 1\}} = \frac{1}{2}(1 \pm \sigma_{v_{i-1}})$, so that the above equals

$$\begin{aligned} &\frac{1}{2}(1 + \sigma_{v_{i-1}}) \frac{e^{\beta + h_{v_i}} - e^{-\beta - h_{v_i}}}{e^{\beta + h_{v_i}} + e^{-\beta - h_{v_i}}} + \frac{1}{2}(1 - \sigma_{v_{i-1}}) \frac{e^{-\beta + h_{v_i}} - e^{\beta - h_{v_i}}}{e^{-\beta + h_{v_i}} + e^{\beta - h_{v_i}}} \\ &= \sigma_{v_{i-1}} \frac{1}{2} \left(\frac{e^{\beta + h_{v_i}} - e^{-\beta - h_{v_i}}}{e^{\beta + h_{v_i}} + e^{-\beta - h_{v_i}}} - \frac{e^{-\beta + h_{v_i}} - e^{\beta - h_{v_i}}}{e^{-\beta + h_{v_i}} + e^{\beta - h_{v_i}}} \right) + \frac{1}{2} \left(\frac{e^{\beta + h_{v_i}} - e^{-\beta - h_{v_i}}}{e^{\beta + h_{v_i}} + e^{-\beta - h_{v_i}}} + \frac{e^{-\beta + h_{v_i}} - e^{\beta - h_{v_i}}}{e^{-\beta + h_{v_i}} + e^{\beta - h_{v_i}}} \right). \end{aligned} \tag{6.14}$$

By pairwise combining the terms over a common denominator the above equals

$$\begin{aligned}\sigma_{v_{i-1}}&\frac{1}{2}\frac{(\mathrm{e}^{\beta+h_{v_i}}-\mathrm{e}^{-\beta-h_{v_i}})(\mathrm{e}^{-\beta+h_{v_i}}+\mathrm{e}^{\beta-h_{v_i}})-(\mathrm{e}^{-\beta+h_{v_i}}-\mathrm{e}^{\beta-h_{v_i}})(\mathrm{e}^{\beta+h_{v_i}}+\mathrm{e}^{-\beta-h_{v_i}})}{(\mathrm{e}^{\beta+h_{v_i}}+\mathrm{e}^{-\beta-h_{v_i}})(\mathrm{e}^{-\beta+h_{v_i}}+\mathrm{e}^{\beta-h_{v_i}})}\\ &+\frac{1}{2}\frac{(\mathrm{e}^{\beta+h_{v_i}}-\mathrm{e}^{-\beta-h_{v_i}})(\mathrm{e}^{-\beta+h_{v_i}}+\mathrm{e}^{\beta-h_{v_i}})+(\mathrm{e}^{-\beta+h_{v_i}}-\mathrm{e}^{\beta-h_{v_i}})(\mathrm{e}^{\beta+h_{v_i}}+\mathrm{e}^{-\beta-h_{v_i}})}{(\mathrm{e}^{\beta+h_{v_i}}+\mathrm{e}^{-\beta-h_{v_i}})(\mathrm{e}^{-\beta+h_{v_i}}+\mathrm{e}^{\beta-h_{v_i}})}. \end{aligned}\tag{6.15}$$

By expanding all products, this equals, after cancellations,

$$\begin{aligned}&\sigma_{v_{i-1}}\frac{\mathrm{e}^{2\beta}+\mathrm{e}^{-2\beta}}{\mathrm{e}^{2\beta}+\mathrm{e}^{-2\beta}+\mathrm{e}^{2h_{v_i}}+\mathrm{e}^{-2h_{v_i}}}+\frac{\mathrm{e}^{2h_{v_i}}+\mathrm{e}^{-2h_{v_i}}}{\mathrm{e}^{2\beta}+\mathrm{e}^{-2\beta}+\mathrm{e}^{2h_{v_i}}+\mathrm{e}^{-2h_{v_i}}}\\ &\quad=\sigma_{v_{i-1}}\frac{\sinh(2\beta)}{\cosh(2\beta)+\cosh(2h_{v_i})}+\frac{\sinh(2h_{v_i})}{\cosh(2\beta)+\cosh(2h_{v_i})}.\end{aligned}\tag{6.16}$$

Using this, we have that

$$\langle\sigma_{v_\ell}\rangle^{t,+/f}=\langle\langle\sigma_{v_\ell}|S_{\le\ell-1}\rangle^{t,+/f}\rangle^{t,+/f}=\langle\sigma_{v_{\ell-1}}\rangle^{t,+/f}\frac{\sinh(2\beta)}{\cosh(2\beta)+\cosh(2h_{v_\ell})}+\frac{\sinh(2h_{v_\ell})}{\cosh(2\beta)+\cosh(2h_{v_\ell})}.\tag{6.17}$$

Applying this recursively, we get

$$\begin{aligned}\langle\sigma_{v_\ell}\rangle^{t,+/f}=\langle\sigma_{v_0}\rangle^{t,+/f}&\prod_{i=1}^{\ell}\frac{\sinh(2\beta)}{\cosh(2\beta)+\cosh(2h_{v_i})}\\ &+\sum_{i=1}^{\ell}\left(\frac{\sinh(2h_{v_i})}{\cosh(2\beta)+\cosh(2h_{v_i})}\prod_{k=i+1}^{\ell}\frac{\sinh(2\beta)}{\cosh(2\beta)+\cosh(2h_{v_k})}\right).\end{aligned}\tag{6.18}$$

Similarly,

$$\begin{aligned}\langle\sigma_{v_0}\sigma_{v_\ell}\rangle^{t,+/f}&=\Bigg\langle\sigma_{v_0}\Bigg(\sigma_{v_0}\prod_{i=1}^{\ell}\frac{\sinh(2\beta)}{\cosh(2\beta)+\cosh(2h_{v_i})}\\ &\qquad+\sum_{i=1}^{\ell}\left(\frac{\sinh(2h_{v_i})}{\cosh(2\beta)+\cosh(2h_{v_i})}\prod_{k=i+1}^{\ell}\frac{\sinh(2\beta)}{\cosh(2\beta)+\cosh(2h_{v_k})}\right)\Bigg)\Bigg\rangle^{t,+/f}\\ &=\prod_{i=1}^{\ell}\frac{\sinh(2\beta)}{\cosh(2\beta)+\cosh(2h_{v_i})}\\ &\qquad+\langle\sigma_{v_0}\rangle^{t,+/f}\sum_{i=1}^{\ell}\left(\frac{\sinh(2h_{v_i})}{\cosh(2\beta)+\cosh(2h_{v_i})}\prod_{k=i+1}^{\ell}\frac{\sinh(2\beta)}{\cosh(2\beta)+\cosh(2h_{v_k})}\right).\end{aligned}\tag{6.19}$$

Combining the above yields

$$\langle\sigma_{v_0}\sigma_{v_\ell}\rangle^{t,+/f}-\langle\sigma_{v_0}\rangle^{t,+/f}\langle\sigma_{v_\ell}\rangle^{t,+/f}=\Big(1-\big(\langle\sigma_{v_0}\rangle^{t,+/f}\big)^2\Big)\prod_{i=1}^{\ell}\frac{\sinh(2\beta)}{\cosh(2\beta)+\cosh(2h_{v_i})}.\tag{6.20}$$

By taking the limit $t \to \infty$, we obtain

$$\chi = \mathbb{E}\left[\sum_{j \in T_\infty} \left(1 - \langle \sigma_{v_0} \rangle^2\right) \prod_{i=1}^{|j|} \frac{\sinh(2\beta)}{\cosh(2\beta) + \cosh(2h_{v_i})}\right]. \tag{6.21}$$

Finally, we can rewrite

$$\frac{\sinh(2\beta)}{\cosh(2\beta) + \cosh(2h_{v_i})} = \frac{2\sinh(\beta)\cosh(\beta)}{2\cosh(\beta)^2 - 1 + \cosh(2h_{v_i})} = \frac{\theta}{1 + \frac{\cosh(2h_{v_i})-1}{2\cosh(\beta)^2}}, \tag{6.22}$$

so that

$$\chi(\beta, B) = \mathbb{E}\left[\left(1 - \langle \sigma_{v_0} \rangle^2\right) \sum_{j \in T_\infty} \theta^{|j|} \prod_{i=1}^{|j|} \left(1 + \frac{\cosh(2h_{v_i}) - 1}{2\cosh(\beta)^2}\right)^{-1}\right]. \tag{6.23}$$

The rewrite in (6.23) is valid for all β and $B > 0$, and provides the starting point for all our results on the susceptibility.

Identification of the susceptibility for $\beta < \beta_c$. We take the limit $B \searrow 0$, for $\beta < \beta_c$, and apply dominated convergence. First of all, all fields h_i converge to zero by the definition of β_c, so we have pointwise convergence. Secondly, $1 + \frac{\cosh(2h_{v_i})-1}{2\cosh(\beta)^2} \geq 1$, so that the random variable in the expectation is bounded from above by $\sum_{j \in T_\infty} \theta^{|j|}$, which has finite expectation as we show below. Thus, by dominated convergence, the above converges to

$$\lim_{B \searrow 0} \chi(\beta, B) = \mathbb{E}\left[\sum_{j \in T_\infty} \theta^{|j|}\right]. \tag{6.24}$$

Denote by Z_ℓ the number of vertices at distance ℓ from the root. Then,

$$\mathbb{E}\left[\sum_{j \in T_\infty} \theta^{|j|}\right] = \mathbb{E}\left[\sum_{\ell=0}^{\infty} Z_\ell \theta^\ell\right] = \sum_{\ell=0}^{\infty} \mathbb{E}[Z_\ell]\theta^\ell, \tag{6.25}$$

because $Z_\ell \geq 0$, a.s. Note that $Z_\ell/(\mathbb{E}[D]\nu^{\ell-1})$ is a martingale, because the offspring of the root has expectation $\mathbb{E}[D]$ and all other vertices have expected offspring ν. Hence,

$$\lim_{B \searrow 0} \chi(\beta, B) = \sum_{\ell=0}^{\infty} \mathbb{E}[Z_\ell]\theta^\ell = 1 + \sum_{\ell=1}^{\infty} \mathbb{E}[D]\nu^{\ell-1}\theta^\ell = 1 + \frac{\mathbb{E}[D]\theta}{1 - \theta\nu}. \tag{6.26}$$

This proves (6.1). We continue to prove (6.2), which follows by using (5.68) and (5.81):

$$1 + \frac{\mathbb{E}[D]\theta}{\nu(1-\theta^2)}(\beta_c - \beta)^{-1} \leq 1 + \frac{\mathbb{E}[D]\theta}{1 - \theta\nu} \leq 1 + \frac{\mathbb{E}[D]\theta}{\nu(1-\theta_c^2)}(\beta_c - \beta)^{-1}. \tag{6.27}$$

□

6.2 Partial results for the critical exponent $\boldsymbol{\gamma}'$

For the supercritical susceptibility, we prove the following lower bound on $\boldsymbol{\gamma}'$:

Proposition 6.2 (Critical exponent $\boldsymbol{\gamma}'$). *For $\tau \in (3,5]$ or $\mathbb{E}[K^3] < \infty$, there exists a $c > 0$ such that*

$$\chi(\beta, B) \geq c(\beta - \beta_c)^{-1}. \tag{6.28}$$

In particular, if $\boldsymbol{\gamma}'$ exists, then

$$\boldsymbol{\gamma}' \geq 1. \tag{6.29}$$

Proof. We start by rewriting the susceptibility in a form that is convenient in the low-temperature phase.

A rewrite of the susceptibility in terms of i.i.d. random variables. For $\beta > \beta_c$ we start from (6.23). We further rewrite

$$\chi(\beta, B) = \sum_{\ell=0}^{\infty} \theta^{\ell} \mathbb{E}\left[(1 - \langle\sigma_{v_0}\rangle^2) \sum_{v_\ell \in T_\infty} \exp\Big\{ - \sum_{i=1}^{\ell} \log\Big(1 + \frac{\cosh(2h_{v_i}) - 1}{2\cosh(\beta)^2}\Big)\Big\}\right]. \tag{6.30}$$

Here, and in the sequel, we use the convention that empty products, arising when $\ell = 0$, equal 1, while empty sums equal 0. Thus, the contribution due to $\ell = 0$ in the above sum equals 1. We write $v_0 = \phi$ and $v_i = a_0 \cdots a_{i-1} \in \mathbb{N}^i$ for $i \geq 1$, so that v_i the a_{i-1}st child of v_{i-1}. Then,

$$\chi(\beta, B) = \sum_{\ell=0}^{\infty} \theta^{\ell} \sum_{a_0,\ldots,a_{\ell-1}} \mathbb{E}\left[(1 - \langle\sigma_{v_0}\rangle^2) \mathbb{1}_{\{v_\ell \in T_\infty\}} \exp\Big\{ - \sum_{i=1}^{\ell} \log\Big(1 + \frac{\cosh(2h_{v_i}) - 1}{2\cosh(\beta)^2}\Big)\Big\}\right]. \tag{6.31}$$

Let K_{v_i} be the number of children of v_i, and condition on $K_{v_i} = k_i$ for every $i \in [0, \ell-1]$, where we abuse notation to write $[0, m] = \{0, \ldots, m\}$. As a result, we obtain that

$$\begin{aligned}\chi(\beta,B) = &\sum_{\ell=0}^{\infty} \theta^{\ell} \sum_{a_0,\ldots,a_{\ell-1}} \sum_{k_0,\ldots,k_{\ell-1}} \mathbb{P}(v_\ell \in T_\infty, K_{v_i} = k_i \ \forall i \in [0, \ell-1]) \\ &\times \mathbb{E}\left[(1 - \langle\sigma_{v_0}\rangle^2) \exp\Big\{ - \sum_{i=1}^{\ell} \log\Big(1 + \frac{\cosh(2h_{v_i}) - 1}{2\cosh(\beta)^2}\Big)\Big\} \mid v_\ell \in T_\infty, K_{v_i} = k_i \ \forall i \in [0, \ell-1]\right].\end{aligned} \tag{6.32}$$

Note that

$$\mathbb{P}(K_{v_i} = k_i \ \forall i \in [0, \ell-1], v_\ell \in T_\infty) = \mathbb{P}(D = k_0) \mathbb{1}_{\{a_0 \leq k_0\}} \prod_{i=1}^{\ell-1} \mathbb{P}(K = k_i) \mathbb{1}_{\{a_i \leq k_i\}}. \tag{6.33}$$

Let $T_{i,j}$ be the tree that descibes all descendants of the jth child of v_i, with the a_ith child removed, and T_ℓ the offspring of v_ℓ. When $v_\ell \in T_\infty$, all information of the tree T_∞ can be encoded in the collection of trees $(T_{i,j})_{j \in [0, K_{v_i}-1], i \in [0, \ell-1]}$ and T_ℓ, together with the sequence $(a_i)_{i=0}^{\ell-1}$. Denote $\vec{T} = \big((T_{i,j})_{j \in [0, K_{v_i}-1], i \in [0,\ell-1]}, T_\ell\big)$. Then, for any collection of trees $\vec{t} = \big((t_{i,j})_{j \in [0,k_i-1], i \in [0,\ell-1]}, t_\ell\big)$,

$$\mathbb{P}(\vec{T} = \vec{t} \mid K_{v_i} = k_i \ \forall i \in [0, \ell-1], v_\ell \in T_\infty) = \mathbb{P}(T = t_\ell) \prod_{(i,j) \in [0,k_i-1] \times [0,\ell-1]} \mathbb{P}(T = t_{i,j}), \tag{6.34}$$

where the law of T is that of a Galton-Watson tree with offspring distribution K. We conclude that

$$\chi(\beta,B)=\sum_{\ell=0}^{\infty}\theta^{\ell}\sum_{a_0,\ldots,a_{\ell-1}}\sum_{k_0,\ldots,k_{\ell-1}}\mathbb{P}(D=k_0)\mathbb{1}_{\{a_0\leq k_0\}}\prod_{i=1}^{\ell-1}\mathbb{P}(K=k_i)\mathbb{1}_{\{a_i\leq k_i\}} \tag{6.35}$$
$$\times\,\mathbb{E}\left[(1-\langle\sigma_{v_0}\rangle^2)\exp\Big\{-\sum_{i=1}^{\ell}\log\Big(1+\frac{\cosh(2h_i^{\star}(\vec{k}))-1}{2\cosh(\beta)^2}\Big)\Big\}\right],$$

where $(h_i^{\star}(\vec{k}))_{i=0}^{\ell}$ satisfy the recursion relations $h_{\ell}^{\star}=h_{\ell,1}$

$$h_i^{\star}(\vec{k})=B+\xi(h_{i+1}^{\star}(\vec{k}))+\sum_{j=1}^{k_i-1}\xi(h_{i,j}), \tag{6.36}$$

and where $(h_{i,j})_{i\in[0,\ell],j\geq 1}$ are i.i.d. copies of the random variable $h(\beta,B)$. We note that the law of $(h_i^{\star}(\vec{k}))_{i=0}^{\ell}$ does not depend on $(a_i)_{i\in[0,\ell-1]}$, so that the summation over $(a_i)_{i\in[0,\ell-1]}$ yields

$$\chi(\beta,B)=\sum_{\ell=0}^{\infty}\theta^{\ell}\sum_{k_0,\ldots,k_{\ell-1}}k_0\mathbb{P}(D=k_0)\prod_{i=1}^{\ell-1}k_i\mathbb{P}(K=k_i) \tag{6.37}$$
$$\times\,\mathbb{E}\left[(1-\langle\sigma_{v_0}\rangle^2)\exp\Big\{-\sum_{i=1}^{\ell}\log\Big(1+\frac{\cosh(2h_i^{\star}(\vec{k}))-1}{2\cosh(\beta)^2}\Big)\Big\}\right].$$

For a random variable X on the non-negative integers with $\mathbb{E}[X]>0$, we let $X^{\star}$ be the size-biased distribution of X given by

$$\mathbb{P}(X^{\star}=k)=\frac{k}{\mathbb{E}[X]}\mathbb{P}(X=k). \tag{6.38}$$

Then

$$\chi(\beta,B)=\frac{\mathbb{E}[D]}{\nu}\sum_{\ell=0}^{\infty}(\theta\nu)^{\ell}\sum_{k_0,\ldots,k_{\ell-1}}\mathbb{P}(D^{\star}=k_0)\prod_{i=1}^{\ell-1}\mathbb{P}(K^{\star}=k_i) \tag{6.39}$$
$$\times\,\mathbb{E}\left[(1-\langle\sigma_{v_0}\rangle^2)\exp\Big\{-\sum_{i=1}^{\ell}\log\Big(1+\frac{\cosh(2h_i^{\star}(\vec{k}))-1}{2\cosh(\beta)^2}\Big)\Big\}\right].$$

Define $(h_i^{\star})_{i=0}^{\ell}=\big(h_i^{\star}(D^{\star},K_1^{\star},\ldots,K_{\ell-1}^{\star},K_{\ell})\big)_{i=0}^{\ell}$, where the random variables $(D^{\star},K_1^{\star},\ldots,K_{\ell-1}^{\star},K_{\ell})$ are independent. Then we finally arrive at

$$\chi(\beta,B)=\frac{\mathbb{E}[D]}{\nu}\sum_{\ell=0}^{\infty}(\theta\nu)^{\ell}\mathbb{E}\left[(1-\langle\sigma_{v_0}\rangle^2)\exp\Big\{-\sum_{i=1}^{\ell}\log\Big(1+\frac{\cosh(2h_i^{\star})-1}{2\cosh(\beta)^2}\Big)\Big\}\right]. \tag{6.40}$$

Reduction to second moments. We now proceed towards the lower bound on $\boldsymbol{\gamma'}$. Note that, a.s.,

$$\langle \sigma_{v_0} \rangle = \tanh(h^\star_{v_0}), \tag{6.41}$$

where

$$h_{v_0^\star} = B + \xi(h^\star_{v_1}) + \sum_{j=1}^{D^\star-1} \xi(h_{0,j}) \leq B + \beta + \sum_{j=1}^{D^\star-1} \xi(h_{0,j}). \tag{6.42}$$

Therefore,

$$\langle \sigma_{v_0} \rangle \leq \tanh(B + \beta + \sum_{j=1}^{D^\star-1} \xi(h_{0,j})). \tag{6.43}$$

The right hand side is independent of $(h^\star_i)_{i=1}^{\ell}$, so that the expectation factorizes. Further,

$$\mathbb{E}\Big[\tanh(B + \beta + \sum_{j=1}^{D^\star-1} \xi(h_{0,j}))\Big] \to \tanh(\beta) = \theta < 1, \tag{6.44}$$

as $B \searrow 0, \beta \searrow \beta_c$. Further, we restrict the sum over all ℓ to $\ell \leq m$, where we take $m = (\beta - \beta_c)^{-1}$. This leads to

$$\chi(\beta, B) \geq \frac{(1-\theta^2)\mathbb{E}[D]}{\nu} \sum_{\ell=0}^{m} (\theta\nu)^\ell \mathbb{E}\left[\exp\Big\{ - \sum_{i=1}^{\ell} \log\Big(1 + \frac{\cosh(2h^\star_i) - 1}{2\cosh(\beta)^2}\Big)\Big\}\right]. \tag{6.45}$$

We condition on all coordinates of $(D^\star, K^\star_1, \ldots, K^\star_{\ell-1}, K_\ell)$ being at most $b = (\beta - \beta_c)^{-1/(\tau-3)}$, which has probability

$$\begin{aligned}\mathbb{P}(D^\star \leq b, K^\star_1 \leq b, \ldots, K^\star_{\ell-1} \leq b, K_\ell \leq b) &\geq (1 - o(1))\mathbb{P}(K^\star \leq b)^m \\ &\geq (1 - o(1))\big(1 - C_{K^\star} b^{-(\tau-3)}\big)^m,\end{aligned} \tag{6.46}$$

which is uniformly bounded from below by a constant for the choices $m = (\beta - \beta_c)^{-1}$ and $b = (\beta - \beta_c)^{-1/(\tau-3)}$. Also, we use that $\theta\nu \geq 1$, since $\beta > \beta_c$. This leads us to

$$\chi(\beta, B) \geq c_\chi \sum_{\ell=0}^{m} \overline{\mathbb{E}}_b\left[\exp\Big\{ - \sum_{i=1}^{\ell} \log\Big(1 + \frac{\cosh(2h^\star_i) - 1}{2\cosh(\beta)^2}\Big)\Big\}\right], \tag{6.47}$$

where $\overline{\mathbb{E}}_b$ denotes the conditional expectation given that $D^\star \leq b, K^\star_1 \leq b, \ldots, K^\star_{\ell-1} \leq b, K_\ell \leq b$. Using that $\mathbb{E}[\mathrm{e}^X] \geq \mathrm{e}^{\mathbb{E}[X]}$, this leads us to

$$\chi(\beta, B) \geq c_\chi \sum_{\ell=0}^{m} \exp\Big\{ - \sum_{i=1}^{\ell} \overline{\mathbb{E}}_b\left[\log\Big(1 + \frac{\cosh(2h^\star_i) - 1}{2\cosh(\beta)^2}\Big)\right]\Big\}. \tag{6.48}$$

Define, for $a > 0$ and $x \geq 0$, the function $q(x) = \log\Big(1 + a(\cosh(x) - 1)\Big)$. Differentiating leads to

$$q'(x) = \frac{a\sinh(x)}{1 + a(\cosh(x) - 1)}, \tag{6.49}$$

so that $q'(x) \leq C_q x/2$ for some constant C_q and all $x \geq 0$. As a result, $q(x) \leq C_q x^2/4$, so that

$$\chi(\beta, B) \geq c_\chi \sum_{\ell=0}^{m} \exp\Big\{ - C_q \sum_{i=1}^{\ell} \overline{\mathbb{E}}_b\left[(h^\star_i)^2\right]\Big\}. \tag{6.50}$$

Second moment analysis of $h_i^\star$. As a result, it suffices to investigate second moments of $h_i^\star$, which we proceed with now. We note that

$$h_i^\star = \xi(h_{i+1}^\star) + B + \sum_{j=1}^{K_i^\star - 1} \xi(h_{i,j}). \tag{6.51}$$

Taking expectations and using that $\xi(h) \leq \theta h$ leads to

$$\overline{\mathbb{E}}_b\left[h_i^\star\right] \leq \theta \overline{\mathbb{E}}_b\left[h_{i+1}^\star\right] + B + \mathbb{E}[K^\star - 1 \mid K^\star \leq b]\mathbb{E}[\xi(h)]. \tag{6.52}$$

Iterating this inequality until $\ell - i$ and using that $\overline{\mathbb{E}}_b\left[h_\ell^\star\right] \leq B + \nu \mathbb{E}[\xi(h)]$ (since $\overline{\mathbb{E}}_b[K] \leq \mathbb{E}[K]$) leads to

$$\begin{aligned}\overline{\mathbb{E}}_b\left[h_i^\star\right] &\leq \theta^{\ell-i}(B + \nu\mathbb{E}[\xi(h)]) + \sum_{s=0}^{\ell-i-1} \theta^s \big(B + \mathbb{E}[K^\star - 1 \mid K^\star \leq b]\mathbb{E}[\xi(h)]\big) \\ &\leq \theta^{\ell-i}(B + \nu\mathbb{E}[\xi(h)]) + \frac{B + \mathbb{E}[K^\star - 1 \mid K^\star \leq b]\mathbb{E}[\xi(h)]}{1-\theta}.\end{aligned} \tag{6.53}$$

Similarly,

$$\begin{aligned}\overline{\mathbb{E}}_b\left[(h_i^\star)^2\right] \leq{}& \theta^2 \overline{\mathbb{E}}_b\left[(h_{i+1}^\star)^2\right] + 2\theta \overline{\mathbb{E}}_b\left[h_{i+1}^\star\right]\big(B + \mathbb{E}[K^\star - 1 \mid K^\star \leq b]\mathbb{E}[\xi(h)]\big) \\ &+ B^2 + 2B\mathbb{E}[K^\star - 1 \mid K^\star \leq b]\mathbb{E}[\xi(h)] + \mathbb{E}[(K^\star - 1)(K^\star - 2) \mid K^\star \leq b]\mathbb{E}[\xi(h)]^2 \\ &+ \mathbb{E}[K^\star - 1 \mid K^\star \leq b]\mathbb{E}[\xi(h)^2].\end{aligned} \tag{6.54}$$

Taking the limit $B \searrow 0$ we thus obtain

$$\begin{aligned}\overline{\mathbb{E}}_b\left[(h_i^\star)^2\right] \leq{}& \theta^2 \overline{\mathbb{E}}_b\left[(h_{i+1}^\star)^2\right] + 2\theta \overline{\mathbb{E}}_b\left[h_{i+1}^\star\right]\mathbb{E}[K^\star - 1 \mid K^\star \leq b]\mathbb{E}[\xi(h)] \\ &+ \mathbb{E}[(K^\star - 1)(K^\star - 2) \mid K^\star \leq b]\mathbb{E}[\xi(h)]^2 + \mathbb{E}[K^\star - 1 \mid K^\star \leq b]\mathbb{E}[\xi(h)^2].\end{aligned} \tag{6.55}$$

We start analysing the case where $\mathbb{E}[K^3] < \infty$. By Theorem 2.8, for $\mathbb{E}[K^3] < \infty$,

$$\mathbb{E}[\xi(h)] \leq C_0(\beta - \beta_c)^{1/2}, \tag{6.56}$$

for some constant C_0. Substituting (6.53), and iterating in a similar fashion as in the proof of (6.53), we obtain that, for $\mathbb{E}[K^3] < \infty$,

$$\overline{\mathbb{E}}_b\left[(h_i^\star)^2\right] \leq C(\beta - \beta_c). \tag{6.57}$$

We next extend this analysis to $\tau \in (3, 5)$. Note that, for every $a > 0$,

$$\mathbb{E}[(K^\star)^a \mid K^\star \leq b] = \frac{\mathbb{E}[K^{a+1}\mathbb{1}_{\{K \leq b\}}]}{\mathbb{E}[K\mathbb{1}_{\{K \leq b\}}]}, \tag{6.58}$$

so that, for $\tau \in (3, 5)$,

$$\mathbb{E}[(K^\star)^2 \mid K^\star \leq b] \leq \frac{C_{3,\tau}}{\mathbb{E}[K\mathbb{1}_{\{K \leq b\}}]} b^{5-\tau}, \tag{6.59}$$

Further, for $\tau \in (3,5)$,

$$\mathbb{E}[\xi(h)] \leq C_0(\beta - \beta_c)^{1/(3-\tau)}, \tag{6.60}$$

and thus

$$\mathbb{E}[(K^\star)^2 \mid K^\star \leq b]\mathbb{E}[\xi(h)]^2 C \leq b^{5-\tau}\mathbb{E}[\xi(h)]^2 \leq C(\beta-\beta_c)^{-(5-\tau)/(3-\tau)+2/(3-\tau)} = C(\beta-\beta_c). \tag{6.61}$$

It can readily be seen that all other contributions to $\overline{\mathbb{E}}_b\left[(h_i^\star)^2\right]$ are of the same or smaller order. For example, when $\mathbb{E}[K^2] < \infty$ and using that $1/(\tau-3) \geq 1/2$ for all $\tau \in (3,5)$,

$$\mathbb{E}[K^\star - 1 \mid K^\star \leq b]\mathbb{E}[\xi(h)^2] \leq C\mathbb{E}[\xi(h)]^2 = O(\beta-\beta_c), \tag{6.62}$$

while, when $\tau \in (3,4)$,

$$\mathbb{E}[K^\star - 1 \mid K^\star \leq b]\mathbb{E}[\xi(h)^2] \leq Cb^{4-\tau}\mathbb{E}[\xi(h)]^{\tau-2} = C(\beta-\beta_c)^{-(4-\tau)/(3-\tau)+(\tau-2)/(3-\tau)} = C(\beta-\beta_c)^2. \tag{6.63}$$

We conclude that

$$\overline{\mathbb{E}}_b\left[(h_i^\star)^2\right] \leq C(\beta-\beta_c). \tag{6.64}$$

Therefore,

$$\chi(\beta,B) \geq c_\chi \sum_{\ell=0}^{m} \exp\Big\{-C\ell(\beta-\beta_c)\Big\} = O((\beta-\beta_c)^{-1}), \tag{6.65}$$

as required.

The proof for $\tau = 5$ is similar when noting that the logarithmic corrections present in $\mathbb{E}[\xi(h)]^2$ and in $\mathbb{E}[(K^\star)^2 \mid K^\star \leq b]$ precisely cancel. $\square$

We close this section by performing a heuristic argument to determine the upper bound on $\boldsymbol{\gamma}'$. Unfortunately, as we will discuss in more detail following the heuristics, we are currently not able to turn this analysis into a rigorous proof.

The upper bound on $\boldsymbol{\gamma}'$: heuristics for $\mathbb{E}[K^3] < \infty$. We can bound from above

$$\chi(\beta,B) \leq \frac{\mathbb{E}[D]}{\nu}\sum_{\ell=0}^{\infty}(\theta\nu)^\ell \mathbb{E}\left[\exp\Big\{-\sum_{i=1}^{\ell}\log\Big(1+\frac{\cosh(2h_i^\star)-1}{2\cosh(\beta)^2}\Big)\Big\}\right]. \tag{6.66}$$

Now, the problem is that $\theta\nu > 1$ when $\beta > \beta_c$, so that we need to extract extra decay from the exponential term, which is technically demanding, and requires us to know various constants rather precisely. Let us show this heuristically. It suffices to study large values of ℓ, since small values can be bounded in a simple way.

We blindly put the expectation in the exponential, and Taylor expand to obtain that

$$\chi(\beta,B) \approx \frac{\mathbb{E}[D]}{\nu}\sum_{\ell=0}^{\infty}(\theta\nu)^\ell \exp\Big\{-\sum_{i=1}^{\ell}\frac{\mathbb{E}\left[(h_i^\star)^2\right]}{\cosh(\beta)^2}\Big\}. \tag{6.67}$$

We compute that

$$\cosh(\beta)^2 = \frac{1}{1-\theta^2}. \tag{6.68}$$

Since

$$h_i^\star \approx \theta h_{i+1}^\star + \sum_{j=1}^{K_i^\star - 1} \xi(h_{i,j}), \tag{6.69}$$

we have

$$\mathbb{E}\left[h_i^\star\right] \approx \frac{\mathbb{E}[K^\star - 1]}{1-\theta}\mathbb{E}[\xi(h)], \tag{6.70}$$

and

$$\mathbb{E}\left[(h_i^\star)^2\right] \approx \frac{2\theta\mathbb{E}[K^\star-1]^2 + \mathbb{E}[(K^\star-1)(K^\star-2)](1-\theta)}{(1-\theta^2)(1-\theta)}\mathbb{E}[\xi(h)]^2 + \frac{\mathbb{E}[K^\star-1]}{1-\theta^2}\mathbb{E}[\xi(h)^2]. \tag{6.71}$$

Ignoring all error terms in the proof of Lemma 5.1 shows that

$$\mathbb{E}[\xi(h)^2] \approx \frac{\nu_2\theta^2}{1-\theta}\mathbb{E}[\xi(h)]^2 = C_2\mathbb{E}[\xi(h)]^2, \tag{6.72}$$

so in total we arrive at (also using that $\theta \approx 1/\nu$)

$$\mathbb{E}\left[(h_i^\star)^2\right] \approx \frac{\nu_3(1-\theta)/\nu + 3\nu_2^2/\nu^3}{(1-\theta^2)(1-\theta)}\mathbb{E}[\xi(h)]^2. \tag{6.73}$$

As a result,

$$\frac{\mathbb{E}\left[(h_i^\star)^2\right]}{\cosh(\beta)^2} \approx \frac{\nu_3(1-\theta)/\nu + 3\nu_2^2/\nu^3}{1-\theta}\mathbb{E}[\xi(h)]^2. \tag{6.74}$$

Ignoring error terms in the computation in Lemma 5.2 shows that

$$\mathbb{E}[\xi(h)^3] \approx C_3\mathbb{E}[\xi(h)]^3, \tag{6.75}$$

where

$$C_3 = \frac{\theta^3}{1-\theta^3\nu}\left(\nu_3 + 3\nu_2 C_2\right) \approx \frac{\theta^3}{1-\theta^2}\left(\nu_3 + 3\nu_2 C_2\right) = \frac{\theta^3}{(1-\theta^2)(1-\theta)}\left(\nu_3(1-\theta) + 3(\nu_2/\nu)^2\right), \tag{6.76}$$

since $\theta \approx 1/\nu$. Further, again ignoring error terms in (5.24) and Taylor expanding to third order shows that

$$\mathbb{E}[\xi(h)] \approx \theta\nu\mathbb{E}[\xi(h)] - C_1\mathbb{E}[\xi(h)]^3, \tag{6.77}$$

where

$$C_1 = -\frac{\xi'''(0)}{6}\big(\nu C_3 + 3\nu_2 C_2 + \nu_3\big), \tag{6.78}$$

and $\xi'''(0) = -2\theta(1-\theta^2)$. Substituting the definitions for C_2 and C_3 yields

$$\begin{aligned} C_1 &= \frac{\theta(1-\theta^2)}{3}\big(\nu C_3 + 3\nu_2 C_2 + \nu_3\big) \\ &= \frac{\theta}{3(1-\theta)}\big(\nu\theta^3\nu_3(1-\theta) + 3\nu\theta^3(\nu_2/\nu)^2 + 3\nu_2^2\theta^2(1-\theta^2) + \nu_3(1-\theta)(1-\theta^2)\big) \\ &= \frac{\theta}{3(1-\theta)}\big(\nu_3(1-\theta) + 3\nu_2^2\theta^2\big). \end{aligned} \tag{6.79}$$

Thus, we arrive at

$$\mathbb{E}[\xi(h)]^2 \approx \frac{\theta\nu - 1}{C_1}, \tag{6.80}$$

so that substitution into (6.74) leads to

$$\frac{\mathbb{E}\left[(h_i^\star)^2\right]}{\cosh(\beta)^2} \approx (\theta\nu - 1)\frac{3\big(\nu_3(1-\theta)/\nu + 3\nu_2^2/\nu^3\big)}{\theta\big(\nu_3(1-\theta) + 3\nu_2^2\theta^2\big)} = 3(\theta\nu - 1). \tag{6.81}$$

We conclude that

$$(\theta\nu)\exp\Big\{-\frac{\mathbb{E}\left[(h_i^\star)^2\right]}{\cosh(\beta)^2}\Big\} \le \big(1+(\theta\nu-1)\big)\mathrm{e}^{-3(\theta\nu-1)} \le \mathrm{e}^{-2(\theta\nu-1)}. \tag{6.82}$$

This suggests that

$$\chi(\beta, B) \le \frac{\mathbb{E}[D]}{\nu}\sum_{\ell=0}^{\infty} \mathrm{e}^{-2\ell(\theta\nu-1)} = O((\theta\nu-1)^{-1}), \tag{6.83}$$

as required. Also, using (6.67), this suggests that

$$\lim_{\beta \searrow \beta_c} (\theta\nu - 1)\chi(\beta, 0^+) = \mathbb{E}[D]/(2\nu), \tag{6.84}$$

where the constant is precisely half the one for the subcritical susceptibility (see (6.1)). It can be seen by an explicit computation that the same factor 1/2 is also present in the same form for the Curie-Weiss model.

Indeed for the Boltzmann-Gibbs measure with Hamiltonian $H_n(\sigma) = -\frac{1}{2n}\sum_{i,j}\sigma_i\sigma_j$ one has $\beta_c = 1$ and a susceptibility $\chi(\beta, 0^+) = 1/(1-\beta)$ for $\beta < \beta_c$, $\chi(\beta, 0^+) = (1-m^2)/(1-\beta(1-m^2))$ with m the non-zero solution of $m = \tanh(\beta m)$ for $\beta > \beta_c$. Expanding this gives $m^2 = 3(\beta-1)(1+o(1))$ for $\beta \searrow 1$ and hence $\chi(\beta, 0^+) = (1+o(1))/(1-\beta(1-3(\beta-1))) = (1+o(1))/(2(\beta-1))$.

It is a non-trivial task to turn the heuristic of this Section into a proof because of several reasons: (a) We need to be able to justify the step where we put expectations in the exponential. While we are dealing with random variables with small means, they are not independent, so this is demanding; (b) We need to know the constants very precisely, as we are using the fact that a positive and negative term cancel in (6.82). The analysis performed in the previous sections does not give optimal control over these constants, so this step also requires substantial work.

The above heuristic does not apply to $\tau \in (3, 5]$. However, the constant in (6.81) is *always* equal to 3, irrespective of the degree distribution. This suggests that also for $\tau \in (3, 5]$, we should have $\boldsymbol{\gamma'} \le 1$.

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