Metastability of the Ising model on random regular graphs at zero temperature

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Introduction

The *Ising model* is one of the most important models in the study of phase transitions.

Studied on random graphs as model for *cooperative behavior*.

Intersting to study the dynamics of this model, especially *metastability*, where the system might be stuck in a “meta”stable configuration for a very long time, before going to the stable configuration.

The model

Random r *regular graph* on n vertices:

Select graph uniformly at random among all simple r regular graph on n vertices.

Ising model on graph $G=(V,E)$:

Spin configurations: $\sigma \in \{-1,+1\}^V$.

Hamiltonian: $H(\sigma) = -\sum_{(i,j)\in E} \sigma_i \sigma_j - h \sum_{i\in V} \sigma_i$.

Gibbs measure: $\mu(\sigma) = \frac{1}{Z_n} e^{-\beta H(\sigma)}$.

Glauber dynamics with *Metropolis* rates:

At each time step select vertex uniformly at random, say vertex i.

Flip spin σ_i with probability

$$\exp(-\beta[\max(0, H(\sigma^i) - H(\sigma))])$$

where σ^i is the configuration σ where spin σ_i is flipped.

$\mathbf{P}_{-1}$ is the probabilty measure of this dynamics starting from all spins being -1.

τ_{+1} is the first time the system reaches the state where all spins are $+1$.

Main result

Theorem [3] For $r\geq 3$ and h small, there exist $C_1, C_2>0$ so that, with high probability,

$$\lim_{\beta\to\infty} \mathbf{P}_{-1}[e^{\beta(r/2-C_1\sqrt{r})n} < \tau_{+1} < e^{\beta(r/2+C_2\sqrt{r})n}] = 1.$$

Proof strategy

Energy landscape

According to [4,5] sufficient to look at energy landscape, specificaly the *communication height*:

$$\Phi(\mathbf{-1},\mathbf{+1}) = \min_\omega \max_{\sigma\in\omega} H(\sigma),$$

where the $\min$ is over all paths ω from $\mathbf{-1}$ to $\mathbf{+1}$ in the configuration space where only one spin is flipped at a time. It turns out that

$$\tau_{+1} \sim \exp(\beta[\Phi(\mathbf{-1},\mathbf{+1}) - H(\mathbf{-1})]).$$

Isoperimetric number

For $A\subseteq V$, $|A|\leq n/2$ let $|\partial_e A|$ be the number of edges between A and A^c. Then, the *isoperimetric number* is given by:

$$i_e = \min_{|A|\leq n/2} \frac{|\partial_e A|}{|A|}.$$

For $r\geq 3$, [1,2] show that, with high probability,

$$r/2 - \sqrt{\log 2}\sqrt{r} \leq i_e \leq r/2 - C\sqrt{r}.$$

We can use this to show that

$$(r/2 - C_1\sqrt{r})n \leq \Phi(\mathbf{-1},\mathbf{+1}) - H(\mathbf{-1}) \leq (r/2+C_2\sqrt{r})n.$$

Note that this implies that the communication height between $\mathbf{-1}$ and $\mathbf{+1}$ is linear in n. This is different from the Ising model on $\mathbb{Z}^d$, where the communication height is constant.

Future work

Future work includes:

Improve bounds on τ_{+1}.

Generalize results to more general degree sequences.

Investigate metastable behavior at positive temperatures.

References

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