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Distances in power-law random graphs

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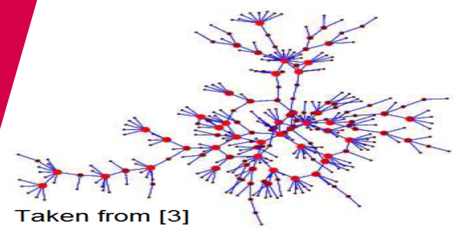

Taken from [3]

Introduction

Empirical studies show that the topology of real-world networks, such as the Internet, social networks and biological networks, have some remarkable similarities [3]:

- *Scale-free* phenomenon:
  The degree sequence of many of these networks obeys a *power law*. This means that, when D_i is the degree of vertex i,
  $$\boldsymbol{P[D_i = k] \sim k^{-\tau},}$$
  for some $\tau > 1$;
- *Small-world* phenomenon:
  Distances in these networks are small compared to the size of the network.

Various random graph models have been proposed to model such real-world networks. We shall focus on:

- the preferential attachment model (*PAM*);
- the configuration model (*CM*).

Random graph models

In the *PAM* the graph grows in time. At each time epoch t a new vertex arrives with a fixed number m edges attached to it. Each of the end points of these edges connect to a vertex i with probability, conditionally on the graph at time $t-1$ and independently of each other, proportional to $D_i(t-1)+\delta$, where $D_i(t)$ is the degree of vertex i at time t and the parameter $\delta > -m$ is introduced to control the power-law exponent. It turns out that the degree sequence in the resulting graph obeys a power law with exponent

$$\boldsymbol{\tau = 3 + \delta/m,}$$

when the number of vertices tends to infinity [2]. Thus, the *PAM* gives a possible explanation for the occurrence of power laws in real-world networks.

For fixed t, the number of half-edges attached to the t vertices in the *CM* are chosen in advance according to a power-law distribution D. The graph is then constructed by pairing the half-edges at random.

It is predicted by physicists that distances in these two models should behave similarly.

Results on distances

All results below hold *with high probability*, which means that the probability the statements are true tends to 1 when the number of vertices tends to ∞.

In the *PAM*, for $m \geq 1$ and $\delta > 0$, i.e. for $\boldsymbol{\tau > 3}$, the diameter of the graph at time t, denoted by $\mathrm{diam}(t)$, satisfies

$$\boldsymbol{c_1 \log t \leq \mathrm{diam}(t) \leq c_2 \log t,}$$

for some constants $c_1, c_2 > 0$ [1]. A similar result holds when $\tau > 3$ in the *CM* [2]. For both models this result can be generalized to bounds on average distances, i.e. the distance between two uniformly chosen connected vertices [1, 2].

In many real-world networks exponents $\boldsymbol{\tau \in (2,3)}$ have been reported [3]. When $m \geq 2$ and $\delta \in (-m, 0)$ in the *PAM*, the diameter satisfies

$$\boldsymbol{c_3 \log\log t \leq \mathrm{diam}(t) \leq c_4 \log\log t,}$$

for some constants $c_3, c_4 > 0$ [1]. The diameter is an upper bound on the average distance, but unfortunately a lower bound on average distances for the *PAM* is not known. For $m = 1$, when the resulting graph is a tree, typical distances are of order $\log t$ [1]. In the *CM*, with $\tau \in (2,3)$, average distances are also of the order $\log\log t$, but the size of the diameter is not necessarily so. When $P[D=1]+P[D=2] > 0$ and $P[D=1] < 1$, the diameter is of order $\log t$ instead [2].

Conclusion

The *small-world* phenomenon is quantified for two random graph models. As shown, the predicted *universality* of distances in power-law random graphs does not always hold for the diameter, since the diameter depends sensitively on the details of the graph. We provide evidence that it does hold for average distances, but some results are still missing and it would be of interest to further investigate this.

References

[1] S. Dommers, R. van der Hofstad, G. Hooghiemstra. *Diameters in preferential attachment models.* arXiv:0705.4153, (2009).

[2] R. van der Hofstad, G. Hooghiemstra. *Universality for distances in random graphs.* J. Math. Phys., 49:125209, (2008).

[3] M.E.J. Newman. *The Structure and Function of Complex Networks.* SIAM Rev., 45(2):167–256, (2003).