A PTAS for the Multiple Depot Vehicle Routing Problem*

Stefan Cardon[1], Sander Dommers[1], Ceyhun Eksin[1], René Sitters[1], André Stougie[3], and Leen Stougie[1,2]

[1] TU Eindhoven
[2] CWI Amsterdam
[3] Hollandse Hout 142, Lelystad, The Netherlands

Abstract. We design a simple PTAS for a multi depot capacitated vehicle routing problem. In this problem a set of customers and set of depots are represented by points in the Euclidean plane. Vehicles have capacities expressed in the number of customers that can be visited on one route starting and ending in a depot. The objective is to determine a set of routes such that all customers are visited and the total length of the routes is minimized. Our results extend the results by Haimovich and Rinnooy Kan [6].

1 Introduction

The basic logistic operation of serving customers with transportation requests by vehicles operated from some depot such as to minimize overall transportation costs is the common theme of *vehicle routing problems*. They belong to the most intensively studied problems in operations research. Most of the research is inspired and directed by practical logistic applications. Surveys of various models and solution methods have appeared in [5], [11] and [12].

The vehicle routing problem which is the subject of this paper is to determine a set of tours of minimum total length such that each of a set of customers, represented by points in the Euclidean plane, is on a tour and such that each tour does not visit more than some given fixed number of customers (the capacity of each vehicle). We assume that all vehicles have the same fixed capacity. The tours are to start and end at a depot. Our results hold for two versions of the problem, one in which each vehicle has to end at the same depot where it starts and the other version where this restriction is not imposed. We notice that the latter version reduces to a single depot problem if the customers are vertices of a graph. However, we consider points in the Euclidean plane for which both versions are relevant.

As a generalisation of the famous travelling salesman problem, in which all customers have unit demand and the vehicle has unlimited capacity, almost all variations of the vehicle routing problem are NP-hard [8] and do not admit high

* This research was partially supported by the Dutch BSIK-BRICKS grant

quality polynomial time approximations. A computer-aided complexity classification of a large class of vehicle routing problems with unit demand of customers has been given in [9]. They highlight some of the intriguing open complexity problems, but do not present approximability and non-approximability results.

Probably the most famous positive result in approximation for routing problems is the celebrated *polynomial time approximation scheme (PTAS)* that Arora [1] devised for the travelling salesman problem in Euclidean finite-dimensional spaces. However, more than a decade earlier, the first such positive result for routing problems was given by Haimovich and Rinnooy Kan [6], who designed a relatively simple PTAS for the capacitated vehicle routing problem described above on the Euclidean plane (see also [7]) in which there is only one depot. Recently, the running time of this PTAS has been improved by Asano *et al.* [2]. Even more recently, some bounds of Haimovich and Rinnooy Kan for VRP have been improved by Bompadre *et al.* [3]. These papers consider problems with a single depot. Charikar et al. [4] derived the strongest constant approximation algorithm for a generalization of the problem, in which items have to be picked-up and delivered ar various points. Their results hold for any metric version of the problem.

We present here, what is to the best of our knowledge, the first PTAS for a multi depot vehicle routing problem. In particular, we show in Section 2 how the PTAS in [6] for the single depot version can be adapted to a PTAS for multi depot versions. In Section 3 we improve the running time considerably by an extension of the results by Asano *et al.* [2] to the multi depot version of the problem.

2 A polynomial time approximation scheme

In the Multiple Depot Vehicle Routing Problem (MDVRP) we are given sets of points $X = \{x_1, \ldots, x_n\}$ (customers) and $Y = \{y_1, \ldots, y_d\}$ (depots) in the Euclidean plane, and a positive integer q. The problem is to find a set of paths that cover all customers and where each path starts and ends in some depot and visits no more than q customers. We distinguish between the problem where the starting point of each path is the same as its end point, and the problem where these points may differ. The goal is to minimize the total length of the paths and we denote the optimal value by $\textsc{Opt}(X)$.

Below we will explicitly analyse the case in which the vehicles have to return to their starting depot and mention in the text where and how the analysis differs in the other case. In fact it turns out that the former case is the difficult one from an approximation point of view. The approximation scheme here is similar to the one in [6] that deals with the single depot problem.

With $\delta(x, y)$ we denote the distance between the two points x and y. Let r_j be the distance of customer j to its nearest depot: $r_j = \min_{i=1,\ldots,d} \delta(x_j, y_i)$. The approximation algorithms have the following structure:

1. Compute the r_j's and relabel the customers $x_1, x_2, \ldots, x_n$ such that $r_1 \geq r_2 \geq \ldots \geq r_n$. Partition X into $X_1, \ldots, X_d$ such that customer j is assigned to X_i only if depot i is the depot nearest to x_j (breaking ties arbitrarily);
2. Define $X(k) := \{x_1, \ldots, x_k\}$, the set of k customers that are furthest away from their nearest depot, and $X \setminus X(k)$, where $k = f(q, d, \varepsilon)$ for some function f (to be specified later);
3. Compute the optimal set of tours for the customers in the set $X(k)$;
4. Apply an approximation algorithm for every depot i and set of customers $X_i \setminus X(k)$.

We prove that this algorithmic scheme is in fact a PTAS if d, the number of depots, and q, the common vehicle capacity, are constants. Thereto we must bound (i) the error that we make by partitioning the set of customers in $X(k)$ and $X \setminus X(k)$ and (ii) the error made by applying an approximation algorithm for customers in $X \setminus X(k)$. We also have to prove that in step 2 the number k can be chosen independently of n. In fact it would be enough if k would not grow too fast with n, so that step 3 can be done in time polynomial in n for fixed d, q and ε.

Lemma 1. *For any* $k \in \{0, 1, \ldots, n\}$,

$$\textsc{Opt}(X(k)) + \textsc{Opt}(X \setminus X(k)) - \textsc{Opt}(X) \leq 2(q-1)kr_k.$$

Proof. Take a circle with radius r_k around each depot. We refer to the customers j with $j \geq k+1$ as *inside points* and the customers j with $j \leq k$ as *outside points.* Consider an optimal solution to the problem. For each tour in this solution that contains an outside point, hence at most $q-1$ inside points, we shortcut the tour in such a way that the vehicle only visits outside points. By the triangle inequality, this does not increase the length of the tour. Each of the remaining inside points on the tour are visited by a separate vehicle, starting from and returning to the nearest depot. Thus, for every inside point a tour of length at most $2r_k$ is added. Hence, for each tour that contains an outside point we make an error of at most $2r_k(q-1)$. This together with the fact that there are k outside points, hence at most k tours in the optimal solution that contain an outside point, proves the lemma. $\square$

In case the vehicles do not have to return to the depot from which they started, this bound can be improved to $4kr_k$ as in [6].

In step 4 one can use any of the approximation algorithms applicable in the PTAS in [6]. We select here the tour partitioning heuristic. It starts with a 2-approximate TSP-tour on $X_i \setminus X(k)$ obtained e.g. by the standard double tree algorithm (see e.g. [10]). Then, following the tour in one of the two directions, chosen arbitrarily, every $(q+1)$-th edge encounterd is deleted, obtaining paths with exactly q customers on it except for possibly the last one. Finally, the end points of these paths are connected to depot i. One can prove ([6]) that the total length of these tours, denoted by $H(X_i \setminus X(k))$, satisfies:

$$H(X_i \setminus X(k)) \leq 2\frac{n_i}{q}\overline{r}^i + c_1 r^i_{\max} + c_2\sqrt{n_i r^i_{\max}\overline{r}^i}\ ,$$

where n_i is the number of customers in $X_i \setminus X(k)$ and $\overline{r}^i$ and $r^i_{\max}$ are the average distance and the maximum distance respectively of the customers in this set. The constants are given by $c_1 = 10 + 4\pi$ and $c_2 = 8\sqrt{\pi}$.

Summing over all depots gives an upper bound on the total length of all tours computed in step 4:

Lemma 2.

$$H(X \setminus X(k)) \leq \frac{2}{q} \sum_{j=k+1}^{n} r_j + dc_1 r_k + c_2 \sum_{i=1}^{d} \sqrt{n_i r_k \overline{r}^i}.$$

□

The following simple lower bound on $\textsc{Opt}(X)$ can be proven in a similar way as in [6]:

Lemma 3.

$$\textsc{Opt}(X) \geq \frac{2}{q} \sum r_j$$

Proof. Let, in the optimal solution, V_h be the set of customers visited by the h-th vehicle and $T^*(V_h)$ the length of the tour serving them. Then

$$T^*(V_h) \geq 2 \max_{x_j \in V_h} \{r_j\} \geq 2 \frac{\sum_{x_j \in V_h} r_j}{|V_h|} \geq 2 \frac{\sum_{x_j \in V_h} r_j}{q},$$

which summed over all h proves the bound. □

Lemma 3 holds for any set X. In particular, we have $\textsc{Opt}(X \setminus X(k)) \geq \frac{2}{q} \sum_{j=k+1}^{n} r_j$.

We are ready to bound the relative error

$$e(k) = \frac{\textsc{Opt}(X(k)) + H(X \setminus X(k)) - \textsc{Opt}(X)}{\textsc{Opt}(X)}.$$

For simplifying notation we write $\sum r_j$ for $\sum_{j=1}^{n} r_j$ and $\overline{r}$ for the average distance $\sum r_j / n$.

Theorem 1.

$$e(k) \leq \left(q(q-1)k + \frac{dqc_1}{2} \right) \frac{r_k}{\sum r_j} + \frac{dqc_2}{2} \sqrt{\frac{r_k}{\sum r_j}}$$

Proof.

$$\begin{aligned}
e(k) &\leq \frac{\textsc{Opt}(X(k)) + H(X \setminus X(k)) + \textsc{Opt}(X \setminus X(k)) - \frac{2}{q}\sum_{j=k+1}^{n} r_j - \textsc{Opt}(X)}{\textsc{Opt}(X)} \\
&\leq \frac{2(q-1)kr_k + \frac{2}{q}\sum_{j=k+1}^{n} r_j + dc_1 r_k + c_2 \sum_{i=1}^{d} \sqrt{n_i r_k \overline{r}^i} - \frac{2}{q}\sum_{j=k+1}^{n} r_j}{\frac{2}{q}\sum r_j} \\
&= \left(q(q-1)k + \frac{dqc_1}{2} \right) \frac{r_k}{\sum r_j} + \frac{qc_2}{2} \frac{\sum_{i=1}^{d} \sqrt{n_i r_k \overline{r}^i}}{\sum r_j} \\
&\leq \left(q(q-1)k + \frac{dqc_1}{2} \right) \frac{r_k}{\sum r_j} + \frac{dqc_2}{2} \sqrt{\frac{r_k}{\sum r_j}}, \qquad (1)
\end{aligned}$$

where we we used Lemma 3 for the first inequality, Lemmas 1 and 2 for the second one, and the fact that $\sqrt{n_i \bar{r}^i}/\sqrt{\sum r_j} \leq 1$ for the last one. $\square$

The next theorem states that a value of k, for which $e(k) \leq \varepsilon$, can be chosen independent of n. The proof is similar to that of the corresponding result in [6].

Theorem 2. *$e(k) \leq \varepsilon$ for*

$$\begin{aligned} k &= \left\lceil \left(1 + \frac{dc_1}{2(q-1)}\right) \left(\exp\left(\frac{q(q-1)}{\varepsilon} + \sqrt{\frac{2c_2^2}{c_1}} \sqrt{\frac{dq}{\varepsilon}}\right) - 1\right) \right\rceil \\ &=: f(d, q, \varepsilon). \end{aligned}$$

Proof. Using (1) we write $e(k) \leq \varepsilon$ as a quadratic equation in $\sqrt{\frac{r_k}{\sum r_j}}$:

$$\left(k + \frac{dc_1}{2(q-1)}\right) \left(\sqrt{\frac{r_k}{\sum r_j}}\right)^2 + \frac{dc_2}{2(q-1)} \sqrt{\frac{r_k}{\sum r_j}} - \frac{\varepsilon}{q(q-1)} \leq 0. \qquad (2)$$

Since we want k to be the smallest integer that satisfies (2), for $k' < k$, $\sqrt{r_{k'}/\sum r_j}$ should be bigger than the positive root of this equation:

$$\sqrt{\frac{r_{k'}}{\sum r_j}} > \frac{-B + \sqrt{B^2 + 4(k' + A)C}}{2(k' + A)},$$

where $A = \frac{dc_1}{2(q-1)}$, $B = \frac{dc_2}{2(q-1)}$ and $C = \frac{\varepsilon}{q(q-1)}$. Therefore

$$\begin{aligned} 1 &\geq \textstyle\sum_{k'=1}^{k-1} \frac{r'_k}{\sum r_j} > \sum_{k'=1}^{k-1} \left(\frac{-B+\sqrt{B^2+4(k'+A)C}}{2(k'+A)}\right)^2 \\ &\geq C \textstyle\sum_{k'=1}^{k-1} \frac{1}{k'+A} - B\sqrt{C} \sum_{k'=1}^{k-1} \frac{1}{(k'+A)^{3/2}}. \end{aligned}$$

We use that

$$\sum_{k'=1}^{k-1} \frac{1}{k' + A} > \int_{1+A}^{k+A} \frac{1}{z} dz = \ln \frac{k + A}{1 + A},$$

and

$$\sum_{h=1}^{k-1} \frac{1}{(h + A)^{3/2}} < \int_{A}^{k-1+A} \frac{1}{z^{3/2}} dz < \frac{2}{\sqrt{A}},$$

to conclude that

$$C \ln \frac{k + A}{1 + A} - B\sqrt{C} \frac{2}{\sqrt{A}} < 1,$$

and thus

$$k - 1 < (1 + A) \left(\exp\left(\frac{1 + 2B\sqrt{C/A}}{C}\right) - 1\right).$$

Now we reinsert the constants:

$$
\begin{aligned}
k-1 &< \left(1+\frac{dc_1}{2(q-1)}\right)\left(\exp\left(\frac{1+2\frac{dc_2}{2(q-1)}\sqrt{\frac{\varepsilon}{q(q-1)}\frac{2(q-1)}{dc_1}}}{\frac{\varepsilon}{q(q-1)}}\right)-1\right)\\
&= \left(1+\frac{dc_1}{2(q-1)}\right)\left(\exp\left(\frac{q(q-1)}{\varepsilon}+\frac{dqc_2}{\varepsilon}\sqrt{\frac{2\varepsilon}{dqc_1}}\right)-1\right)\\
&= \left(1+\frac{dc_1}{2(q-1)}\right)\left(\exp\left(\frac{q(q-1)}{\varepsilon}+\sqrt{\frac{2c_2^2}{c_1}}\sqrt{\frac{dq}{\varepsilon}}\right)-1\right).
\end{aligned}
$$

Since $k-1$ is the largest integer for which (2) is not true, we can take

$$
k=\left\lceil\left(1+\frac{dc_1}{2(q-1)}\right)\left(\exp\left(\frac{q(q-1)}{\varepsilon}+\sqrt{\frac{2c_2^2}{c_1}}\sqrt{\frac{dq}{\varepsilon}}\right)-1\right)\right\rceil .
$$

to make sure that $e(k)\leq\varepsilon$. □

As a last part of the PTAS proof we analyse the running time. In the first step of the algorithm all distances have to be computed which takes time $O(dn)$ and then have to be sorted, which takes time $O(n\log n)$. Step 2 can be done in constant time assuming d, q and ϵ to be constants. Computing the optimal solution in step 3 takes $O(k^q 2^k)$ time using dynamic programming as follows. For any subset S of $X(k)$ store the length $\textsc{Opt}(S)$ of the optimal solution. If $|S|\leq q$ then its value is computed in $O(2^q)$ time. For larger sets we use $\textsc{Opt}(S)=\min\{\textsc{Opt}(S')+\textsc{Opt}(S\setminus S')\mid S'\subset S, |S'|\leq q\}$. Computing one value takes $O(k^q)$ time. Step 4 of the algorithm takes no more than $O(n_i\log n_i)$ time for every depot i (see e.g. [10]), summing up to $O(n\log n)$. The total running time of the algorithm is

$$
O\left(dn+n\log n+2^{\frac{d}{q}2^{O\left(\frac{q^2}{\varepsilon}+\sqrt{\frac{dq}{\varepsilon}}\right)}}\right).
$$

It should be noted, that in case the vehicles don't have to return to the depot from which they started, the q^2 can be replaced by q in the expression above. This because of the stronger bound of Lemma 1.

3 Improving the running time

In this section, we use a similar approach as in [2] to get rid of the double exponential term in the running time, caused by enumerating the optimal tours covering outside points $X(k)$. First, we review some bounds from [6] and [2].

The iterated tour partitioning algorithm (ITP) is defined as follows. Let T be some TSP-tour on a set of points, W say. As in the tour partitioning heuristic described in the previous section, the tour is partitioned into parts of q points (and possibly one smaller part) and the endpoints of each part are connected with the depot. However, here we do this q times, where each time the partition is shifted one step and take the smallest solution found. In this section we will refer to feasible tours, i.e., tours with at most q customers and depots at the start and the end, as q-tours.

Lemma 4 ([6]). *The length of the ITP-solution is at most* $\frac{2}{q}\sum_{j\in W} r_j + (1 - 1/q)|T|$.

Lemma 5 ([2]). *Given a set* X *of* n *points in the Euclidean plane. let* $r_{max} = \max\{r_1, r_2, \ldots, r_n\}$*. The length of an optimal TSP tour on* X *is bounded by*

$$TSP(X) \leq 10\sqrt{r_{max}\sum_{j\in X} r_j}.$$

The main idea in improving the running time is to relax the partition in inside and outside points. For a set of points U with $X(k) \subseteq U \subseteq X$ we define,

$$Z(X,U) = \textsc{Opt}(U) + \frac{2}{q}\sum_{i\in X\setminus U} r_i. \tag{3}$$

The objective is to find a set $U \supseteq X(k)$ that minimizes $Z(X,U)$. Denote this problem by $\mathcal{P}$ and the minimum value of an instance by $\textsc{Opt}(X, X(k))$. □

Lemma 6.

$$\textsc{Opt}(X, X(k)) \leq \textsc{Opt}(X). \tag{4}$$

Proof. Let T be an optimal set of tours of X and let U be the set of points covered by the q-tours of T that contain at least one outside point. Clearly, $U \supseteq X(k)$ and

$$\textsc{Opt}(X) = \textsc{Opt}(U) + \textsc{Opt}(X \setminus U) \geq \textsc{Opt}(U) + \frac{2}{q}\sum_{j\in X\setminus U} r_j = Z(X,U),$$

where the inequality follows from applying Lemma 3 to the pointset $X \setminus U$. □

Later we show how to compute a solution U for $\mathcal{P}$ with a corresponding set of tours on U. We use ITP for the points not covered by U, i.e., we apply ITP on each of the depots Y_i with the pointset $X_i \setminus U$. For constructing the TSP-tours we use the 2-approximate double-tree solution. Thus, given an optimal solution U, the length of the constructed solution is $\textsc{Opt}(U)$ plus the cost of ITP on each of the depots, which is bounded from above

$$\begin{aligned}\textsc{Opt}(U) + \sum_{i=1}^{d}\left(\frac{2}{q}\sum_{j\in X_i\setminus U} r_j + 2\left(1-\frac{1}{q}\right)TSP(X_i\setminus U)\right)\\ = \textsc{Opt}(X, X(k)) + 2(1-\frac{1}{q})\sum_{i=1}^{d} TSP(X_i \setminus U). \end{aligned}\tag{5}$$

Let $r_\varepsilon(X,d) = \varepsilon^2(10qd)^{-2}\sum_{j\in X} r_j$.

Lemma 7. *For every k with $r_k \leq r_\varepsilon(X,d)$ and $i \in \{1,\ldots,d\}$,*

$$TSP(X_i \setminus U) \leq \frac{\varepsilon}{2d}\textsc{Opt}(X, X(k)). \tag{6}$$

Proof. We use Lemma 5 on the point set $X_i \setminus U$. For any point in this set its distance to depot Y_i is at most $r_k \leq r_\varepsilon(X,d)$.

$$\begin{aligned} TSP(X_i \setminus U) &\leq 10\left(r_k \textstyle\sum_{j\in X_i\setminus U} r_j\right)^{1/2} \\ &\leq 10\left(r_k \textstyle\sum_{j\in X} r_j\right)^{1/2} \\ &\leq 10\left(\varepsilon^2(10qd)^{-2}(\textstyle\sum_{j\in X} r_j)^2\right)^{1/2} \\ &= \tfrac{\varepsilon}{2d}\tfrac{2}{q}\textstyle\sum_{j\in X} r_j \\ &\leq \tfrac{\varepsilon}{2d}\textsc{Opt}(X, X(k)). \end{aligned}$$

□

Using Lemmas 6 and 7, the total length, given in (5), is bounded by

$$\begin{aligned} \textsc{Opt}(X,X(k)) + 2(1-\tfrac{1}{q})\textstyle\sum_{i=1}^{d}\tfrac{\varepsilon}{2d}\textsc{Opt}(X,X(k)) &< (1+\varepsilon)\textsc{Opt}(X,X(k)) \\ &\leq (1+\varepsilon)\textsc{Opt}(X). \end{aligned}$$

Next we show how to get a $(1+\varepsilon)$-approximate solution U for $\mathcal{P}$. Then, the total approximation factor will become $(1+\varepsilon)^2$. The grid approach used in [2] (See also [1]) for the single depot problem applies here with little adjustment. Let W be an optimal solution for $\mathcal{P}$. We may assume that each tour in a corresponding set of optimal tours contains at least one point from $X(k)$. Hence, we may assume $|W| \leq qk$. Now we take a grid of $(qk/\varepsilon)\times(qk/\varepsilon)$ points and move each point in W to its nearest grid point. Then one can easily show that the extra cost is at most a factor $1/\epsilon$ of the total length.

We call two solutions U_1 and U_2 equivalent if they round to the same multiset of grid points and a maximal set of equivalent solutions form an equivalence class.

For any solution U we approximate the value $Z(X,U)$ by taking the optimal set of tours through the rounded solution but using the unrounded points for the second term of $Z(X,U)$. For any equivalence class $\mathcal{U}$ we can easily find a solution U that minimizes this approximate value within this class as follows. Since the first term in $Z(X,U)$ is the same within the class we only need to consider the second term. For a point x_j let $\psi(x_j)$ be the grid point to which it is rounded. Suppose that the cardinality of some grid point g in the multiset $\mathcal{U}$ is t and assume $\psi^{-1}(g)\cap X(k) = s$. Then we add the $t-s$ points from $\psi^{-1}(g)\setminus X(k)$ with largest value r_j to U. Further, we add all points $X(k)$.

Now we simply enumerate over all equivalence classes and within each equivalence class $\mathcal{U}$ we enumerate over all possible sets of tours. A tour may be encoded as an ordering of $\mathcal{U}$ together with a vector in $\{1,2,\ldots,d\}^{|\mathcal{U}|}$ indicating the depots used. Thus, the minimum over all equivalence classes can be found in $O\left(((qk/\varepsilon)^2)^{qk} d^{qk}\right)$ time.

Assuming k satisfies $r_k \leq r_\varepsilon(X, d)$ we must have $k \leq (10qd)^2/\varepsilon^2$. Substituting this upper bound on k we obtain a total time of $(qd/\varepsilon)^{O(q^3 d^2/\varepsilon^2)}$. Iterated tour partitioning takes $O(n \log n)$ time.

Theorem 3. *There is a PTAS for MDVRP with running time* $(qd/\varepsilon)^{O(q^3 d^2/\varepsilon^2)} + O(n \log n)$.

4 Epilogue

We constructed a PTAS for a multi depot capacitated vehicle routing problem. The basic PTAS is rather simple and is obtained by adapting the ideas in [6] to the multi depot situation. The improvements that we obtained in the previous section are based on similar ideas for the single depot version in [2].

From the expression of the running time in Theorem 3 we see that the PTAS also holds if $q = O(\log n)$ and $d = O(\log n)$. It would be interesting to find out how far these functions for q and d can be pushed such that the problem remains in the class PTAS. For which functions for q and d will the problem become APX-hard?

References

1. S. Arora, Polynomial Time Approximation Schemes for Euclidean TSP and other Geometric Problems, *Proc. 37th Annual IEEE Symposium on the Foundation of Computer Science*, 2–12, 1996.

2. T. Asano, N. Katoh, H. Tamaki, T. Tokuyama, Covering points inthe plane by k-tours: towards a polynomial time approximation scheme for general k, *Proceedings of the twenty-ninth annual ACM symposium on Theory of computing*, 275–283, 1997.

3. A. Bompadre, M. Dror, J. B. Orlin, Improved bounds for vehicle routing solutions, *Discrete Optimization*, Vol. 3, no. 4, 299–316, 2006.

4. M. Charikar, S. Khuller, B. Raghavachari, Algorithms for Capacitated Vehicle Routing, *SIAM Journal on Computing*, Vol. 31, no. 3, 665–682, 2001.

5. B.L. Golden, A.A. Assad (eds.), *Vehicle Routing: Methods and Studies*, North-Holland, Amsterdam, 1988.

6. M. Haimovich and A.H.G. Rinnooy Kan, Bounds and Heuristics for Capacitated Routing Problems, *Mathematics of Operations Research*, Vol. 10, no. 4, 527–542, 1985.

7. M. Haimovich, A.H.G. Rinnooy Kan, L. Stougie, Analysis of heuristics for vehicle routing problems; B.L. Golden, A.A. Assad (eds.), *Vehicle Routing: Methods and Studies*, North-Holland, Amsterdam, 47–61, 1988.

8. J.K. Lenstra, A.H.G. Rinnooy Kan, Complexity of vehicle routing and scheduling problems, *Networks*, Vol 11, 221–227, 1981

9. W.E. de Paepe, J.K. Lenstra, J. Sgall, R.A. Sitters, L. Stougie, Computer-aided complexity classification of dial-a-ride problems, *INFORMS Journal on Computing* Vol. 16, 120–132, 2004.

10. D.J. Rosenkrantz, R.E. Stearns, P.M. Lewis, An analysis of several heuristics for the traveling salesman problem, *SIAM Journal on Computing*, Vol. 6, no. 3, 563–581, 1977.

11. P. Toth, D. Vigo, *The Vehicle Routing Problem*, SIAM, Philadelphia, 2001.

12. http://neo.lcc.uma.es/radi-aeb/WebVRP/