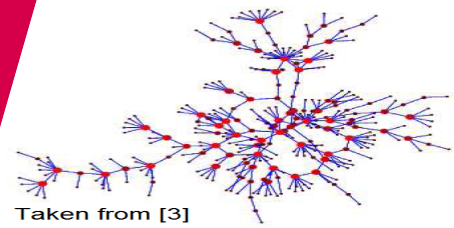
Technische Universiteit Eindhoven University of Technology

Ising models on power-law random graphs

Sander Dommers



Taken from [3]

Introduction

We study the behavior of the *Ising model* on *random graph* models for *complex networks.* There are many real-world examples of complex networks, such as the Internet, social networks and biological networks [3]. Many of these complex networks are reported to have a *power-law* degree distribution. In the model this is reflected by the fact that the probability that vertex i has degree D_i satisfies

$$P[D_i = k] \sim k^{-\tau},$$

for some $\tau > 1$.

There has been much interest in the functionality of such networks in recent years. One interesting model to study on such networks is the Ising model, which is a paradigm model in statistical physics for *cooperative behavior.*

The model

Various random graph models have been proposed to model real-world networks. We shall focus on the *configuration model* (*CM*). For fixed n, the number of half-edges attached to the n vertices in the *CM* are chosen i.i.d. in advance according to a power-law distribution D. The graph is then constructed by pairing the half-edges uniformly at random.

On a graph G_n, the *ferromagnetic Ising model* is given by the following Boltzmann distributions over $\sigma \in \{-1,+1\}^n$,

$$\mu(\sigma) = \frac{1}{Z_n} \exp\left\{\beta \sum_{(i,j)\in E_n} \sigma_i\sigma_j + B \sum_{i\in[n]} \sigma_i\right\},$$

where

- $\beta \geq 0$ is the inverse temperature;
- B is the external magnetic field;
- Z_n is a normalization factor *(partition function).*

Thermodynamic limit of pressure

In [2] we give an explicit expression for the *pressure* per particle

$$\psi_n(\beta, B) = \frac{1}{n} \log Z_n(\beta, B),$$

in the *thermodynamic limit* of $n \to \infty$ when $\tau > 2$, i.e., the degree distribution has *finite mean.* This is a generalization of [1], where the case $\tau > 3$ *(finite variance)* is studied.

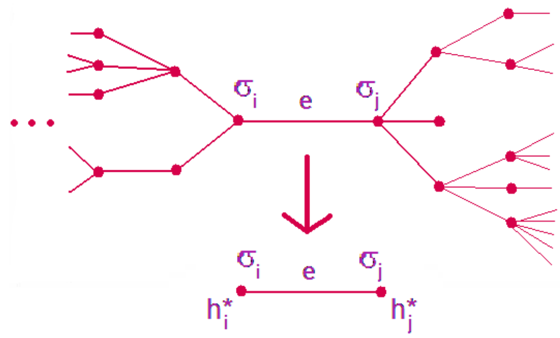
One important property of the *CM* is that the local neighborhood of a randomly chosen edge looks like a *homogeneous tree* where the offspring has *size-biased* degree distribution K, i.e.,

$$\mathbb{P}[K = k] = \frac{(k+1)\mathbb{P}[D = k+1]}{\mathbb{E}[D]}.$$

The Ising model on a tree is simpler to analyse, because the *effective field* a vertex feels can be expressed in terms of that of its neighbors via the *distributional recursion*

$$h^{(t+1)} \stackrel{d}{=} B + \sum_{i=1}^{K_t} \operatorname{atanh}(\tanh(\beta)\tanh(h_i^{(t)})).$$

An edge e and its local neighborhood can thus be replaced by a single edge with external fields distributed as h^*, where h^* is the *fixed point* of the recursion above. This is depicted in the following picture:



Using this, the *internal energy,*

$$-\frac{1}{n} \sum_{(i,j)\in E_n} \langle \sigma_i\sigma_j \rangle_\mu,$$

can easily be computed. Integrating this over β then gives the pressure.

Extensions

Our methodology also allows us to compute other physical quantities such as the *magnetization* and *susceptibility.* In a later paper we will also investigate the *critical behavior* of this model, using the results above.

References

[1] A. Dembo, A. Montanari. *Ising models on locally tree-like graphs.* Ann. Appl. Prob., 20:565–592, (2010).

[2] S. Dommers, C. Giardinà, R. van der Hofstad. *Ising models on power-law random graphs.* Preprint, arXiv:1005.4556, (2010).

[3] M.E.J. Newman. *The structure and function of complex networks.* SIAM Rev., 45:167–256, (2003).