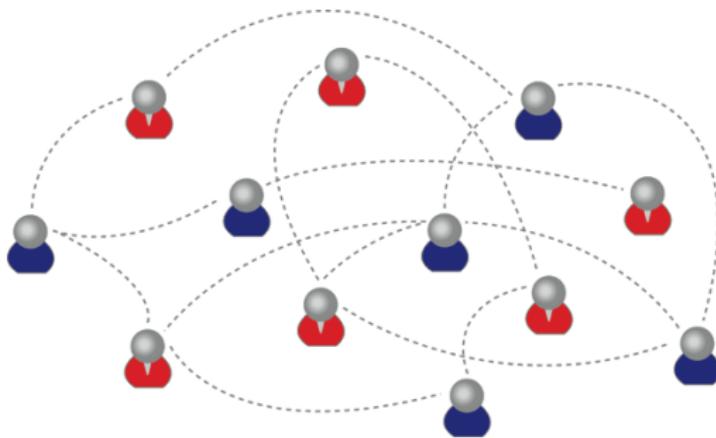


Metastability of the Ising model on random regular graphs at zero temperature

Sander Dommers

Ising model on random graphs

We model opinion spreading with the *Ising model*, a paradigm model in statistical physics for *cooperative behavior*.



What are effects of *structure* of the network on *behavior* of Ising model?

Random regular graphs

Let $r, n \in \mathbb{N}$ such that nr is even

A *random r -regular graph* with n vertices is a graph selected
uniformly at random

from the set of all simple r -regular graphs with n vertices

Denote this random graph by $G_n = (\{1, \dots, n\}, E_n)$

Ising measure on G_n for $\sigma \in \{-1, +1\}^n$

$$\mu(\sigma) = \frac{1}{Z_n} e^{-\beta H(\sigma)}$$

with Hamiltonian

$$H(\sigma) = - \sum_{(i,j) \in E_n} \sigma_i \sigma_j - h \sum_{i=1}^n \sigma_i$$

where

$\beta \geq 0$ inverse temperature

$h > 0$ external magnetic field

Z_n normalization factor (partition function)

Discrete Glauber dynamics

Discrete time *Markov chain* where at every time step

- 1) Pick vertex i uniformly from $\{1, \dots, n\}$

- 2) Flip spin σ_i with probability
$$\begin{cases} 1 & \text{if } H(\sigma^i) \leq H(\sigma) \\ e^{-\beta[H(\sigma^i) - H(\sigma)]} & \text{if } H(\sigma^i) > H(\sigma) \end{cases}$$

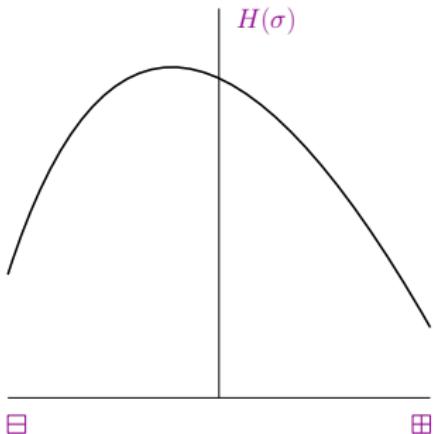
Equilibrium distribution $\mu(\sigma) = \frac{1}{Z_n} e^{-\beta H(\sigma)}$

Metastability

Take zero temperature limit $\beta \rightarrow \infty$

Then, $\mu(\sigma) = \frac{1}{Z_n} e^{-\beta H(\sigma)}$ concentrates on minimizer of

$$H(\sigma) = - \sum_{(i,j) \in E_n} \sigma_i \sigma_j - h \sum_{i=1}^n \sigma_i$$



Minimizer is \blacksquare \rightarrow *stable* state

If h small, local minimum \square
 \rightarrow *metastable* state

Time $\tau(\blacksquare, \square)$ to go from \blacksquare to \square ?

Main result

Theorem (D., 2017)

For random r -regular graphs with $r \geq 3$, h small, whp, there exist constants $0 < C_1 < \infty$, $C_2 < \infty$ such that

$$\lim_{\beta \rightarrow \infty} \mathbb{P}[e^{\beta(r/2 - C_1\sqrt{r})n} < \tau(\square, \boxplus) < e^{\beta(r/2 + C_2\sqrt{r})n}] = 1$$

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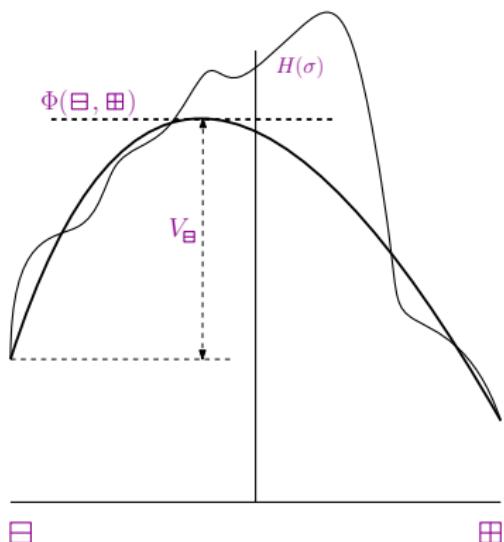
$$\lim_{\beta \rightarrow \infty} \mathbb{P}[e^{\beta(r/2 - C_1\sqrt{r})n} < \tau(\boxminus, \boxplus) < e^{\beta(r/2 + C_2\sqrt{r})n}] = 1$$

Note that exponent is linear in n

Big difference with e.g. finite lattices where $\tau(\boxminus, \boxplus) \sim e^{\beta C}$

Pathwise approach

Cassandro, Galves, Olivieri, Vares, 1984, Neves, Schonmann, 1991
Manzo, Nardi, Olivieri, Scoppola, 2004, Cirillo, Nardi 2013



Communication height

$$\Phi(\sigma, \sigma') = \min_{\omega \text{ path from } \sigma \text{ to } \sigma'} \max_{\sigma'' \in \omega} H(\sigma'')$$

Stability level

$$V_\sigma = \min_{\sigma': H(\sigma') < H(\sigma)} \Phi(\sigma, \sigma') - H(\sigma)$$

Proposition

If there exist $0 < \Gamma_\ell \leq \Gamma_u < \infty$ such that

$$1) \Phi(\square, \blacksquare) - H(\square) \geq \Gamma_\ell$$

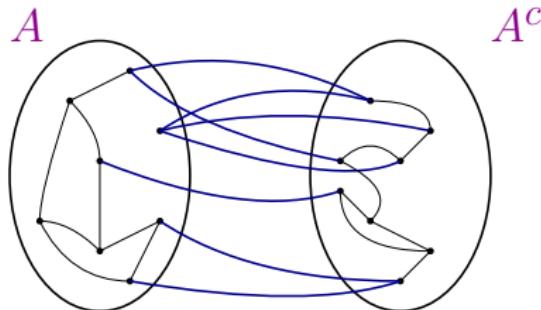
$$2) \Phi(\square, \blacksquare) - H(\square) \leq \Gamma_u$$

$$3) \text{for all } \sigma \notin \{\square, \blacksquare\} \text{ it holds that } V_\sigma \leq \Gamma_u$$

then, for all $\varepsilon > 0$,

$$\lim_{\beta \rightarrow \infty} \mathbb{P}[e^{\beta(\Gamma_\ell - \varepsilon)} < \tau(\square, \blacksquare) < e^{\beta(\Gamma_u + \varepsilon)}] = 1$$

Isoperimetric number

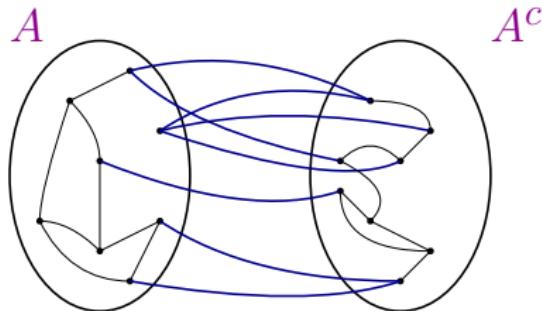


(Edge) isoperimetric number

$$i_e(G_N) = \min_{\substack{A \subset \{1, \dots, n\} \\ |A| \leq n/2}} \frac{|\partial_e A|}{|A|}$$

$$(|\partial_e A| \geq i_e(G_N)|A|)$$

Isoperimetric number



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For r -regular random graphs $\exists C > 0$ such that, whp,
(Bollobás, 1988, Alon, 1997)

$$\frac{r}{2} - \sqrt{\log 2} \sqrt{r} \leq i_e(G_n) \leq \frac{r}{2} - C \sqrt{r}$$

(For lattices $i_e \rightarrow 0$ as $n \rightarrow \infty$)

Lemma

If h small, then $\exists C_1 > 0$ such that $\forall \sigma$ with $n/2 +$ spins

$$H(\sigma) - H(\square) \geq (i_e(G_n) - h)n \geq \left(\frac{r}{2} - C_1\sqrt{r} \right) n$$

Lemma

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$$H(\sigma) - H(\square) \geq (i_e(G_n) - h)n \geq \left(\frac{r}{2} - C_1\sqrt{r}\right)n$$

Proof.

$$H(\square) = - \sum_{(i,j) \in E_n} (-1)^2 - h \sum_{i=1}^n (-1) = -|E_n| + h n$$

With A set of $+$ spins

$$H(\sigma) = -|E_n| + 2|\partial_e A| - \frac{n}{2}h + \frac{n}{2}h \geq 2i_e(G_n)\frac{n}{2} - |E_n|$$

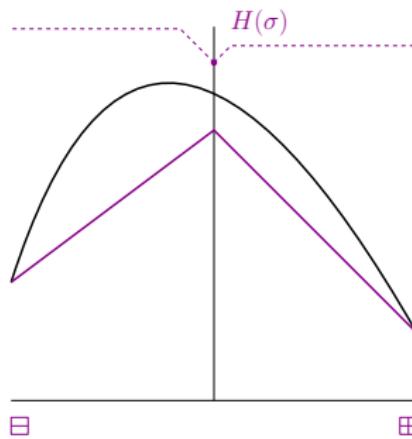
□

Upper bound

For the upper bound use that there exists a configuration with $n/2$ with

$$|\partial_e A| \leq \left(\frac{r}{2} - C\sqrt{r}\right) \frac{n}{2}$$

and that there always exists a path to \square and \blacksquare that doesn't increase energy too much

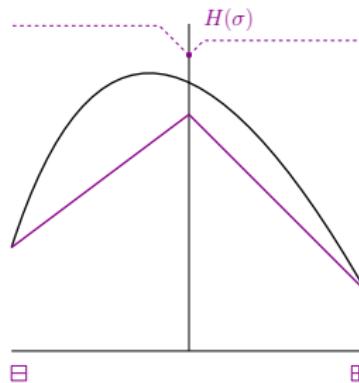


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Extension to more general degrees

Theorem (D., den Hollander, Jovanovski, Nardi, 2017+)

Let all degrees be at least 3. Then, whp, $\exists 0 < \gamma_\ell \leq \gamma_u < \infty$ such that

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Open problems

Prove that $\tau(\boxminus, \boxplus) \sim \exp\{\beta \Gamma_n\}$ with $\lim_{n \rightarrow \infty} \frac{\Gamma_n}{n} = \gamma$

Analyze metastability for $\beta_c < \beta < \infty$ in thermodynamic limit $n \rightarrow \infty$