Ising critical exponents on power-law random graphs

Sander Dommers

Joint work with:
Cristian Giardinà
Remco van der Hofstad

Technische Universiteit
Eindhoven
University of Technology

Disorder in Probability and Statistical Mechanics
June 28, 2012

Ising model

Ising model: paradigm model in statistical physics for *cooperative behavior.*

When studied on complex networks it can model for example *opinion spreading* in society.

We will model complex networks with *power-law random graphs.*

What are effects of *structure* of complex networks on *behavior* of Ising model?

Power-law random graphs

In the *configuration model (CM)* a graph $G_n = (V_n = [n], E_n)$ is constructed as follows.

- Let D have a certain distribution (the *degree distribution*);
- *Assign* D_i half-edges to each vertex $i \in [n]$, where D_i are i.i.d. like D (Add one half-edge to last vertex when the total number of half-edges is odd);
- *Attach* first half-edge to another half-edge *uniformly at random;*
- Continue until all half-edges are connected.

Special attention to power-law degree sequences, i.e.,

$$ck^{-\tau} \leq \mathbb{P}[D = k] \leq Ck^{-\tau}, \qquad \tau > 2.$$

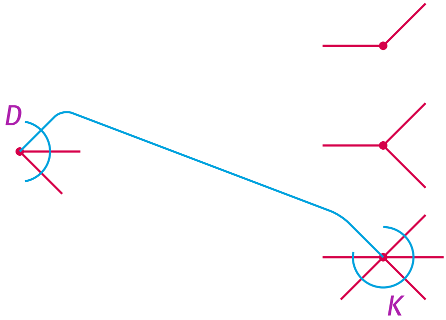
TU/e Technische Universiteit Eindhoven University of Technology

Local structure configuration model for $\tau > 2$

Start from random vertex, which has degree distributed as D, and look at its neighbors.

Local structure configuration model for $\tau > 2$

Start from random vertex, which has degree distributed as D, and look at its neighbors.



Locally tree-like structure: a branching process with offspring D in first generation and K in further generations. Also, *uniformly sparse.*

Definition of the Ising model

On a graph G_n, the *ferromagnetic Ising model* is given by the following Boltzmann distribution over $\sigma \in \{-1, +1\}^n$,

$$\mu(\sigma) = \frac{1}{Z_n(\beta, B)} \exp\left\{\beta \sum_{(i,j)\in E_n} \sigma_i\sigma_j + B \sum_{i\in[n]} \sigma_i\right\},$$

where

- $\beta \geq 0$ is the inverse temperature;
- B is the external magnetic field;
- $Z_n(\beta, B)$ is a normalization factor (the *partition function*), i.e.,

$$Z_n(\beta, B) = \sum_{\sigma\in\{-1,1\}^n} \exp\left\{\beta \sum_{(i,j)\in E_n} \sigma_i\sigma_j + B \sum_{i\in[n]} \sigma_i\right\}.$$

Previous results

Theorem (Dembo, Montanari, '10)

If $\mathbb{E}[K] < \infty$, *then the* pressure *per particle in the thermodynamic limit, a.s.,*

$$\lim_{n\to\infty} \frac{1}{n} \log Z_n(\beta, B) = \varphi(\beta, B),$$

for some explicit function $\varphi(\beta, B)$.

Previous results

Theorem (Dembo, Montanari, '10)

If $\mathbb{E}[K] < \infty$, *then the* pressure *per particle in the thermodynamic limit, a.s.,*

$$\lim_{n\to\infty} \frac{1}{n} \log Z_n(\beta, B) = \varphi(\beta, B),$$

for some explicit function $\varphi(\beta, B)$.

Theorem (DGvdH, '10)

The same holds for $\tau > 2$.

Magnetization

Define the *magnetization* as

$$M(\beta, B) \equiv \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \langle \sigma_i \rangle_\mu,$$

where $\langle \cdot \rangle_\mu$ denotes the expectation under the Ising measure μ.

The *spontaneous magnetization* is then defined as

$$M(\beta, 0^+) \equiv \lim_{B\searrow 0} M(\beta, B).$$

The *critical temperature* β_c equals

$$\beta_c \equiv \inf\{\beta : M(\beta, 0^+) > 0\}.$$

Critical temperature

Theorem (Lyons, '89, DGvdH, '12)

The critical temperature β_c equals, a.s.,

$$\beta_c = \mathrm{atanh}(1/\mathbb{E}[K]).$$

Note that, for $\tau \in (2,3)$, we have $\mathbb{E}[K] = \infty$, so that $\beta_c = 0$.

We study *critical exponents* for $\tau > 3$.

Critical exponents

The *critical exponents* are defined as:

$$M(\beta, 0^+) \asymp (\beta - \beta_c)^{\boldsymbol{\beta}}, \qquad \text{for } \beta \searrow \beta_c;$$
$$M(\beta_c, B) \asymp B^{1/\boldsymbol{\delta}}, \qquad \text{for } B \searrow 0;$$
$$\chi(\beta, 0^+) \asymp (\beta - \beta_c)^{-\boldsymbol{\gamma}}, \qquad \text{for } \beta \nearrow \beta_c,$$

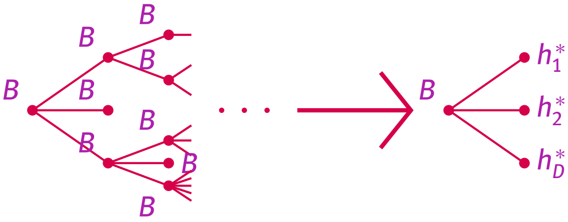
where $\chi(\beta, B) = \frac{\partial}{\partial B} M(\beta, B)$.

Theorem (DGvdH, '12)

	$\mathbb{E}[K^3] < \infty$	$\tau \in (3,4) \cup (4,5)$
$\boldsymbol{\beta}$	$1/2$	$1/(\tau - 3)$
$\boldsymbol{\delta}$	3	$\tau - 2$
$\boldsymbol{\gamma}$	1	1

Tree recursion

Root *magnetization* on a tree:



Effective field h^* is *unique* solution to recursion

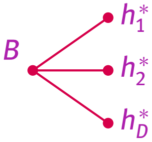
$$h^{(t+1)} \stackrel{d}{=} B + \sum_{i=1}^{K_t} \xi(h_i^{(t)}),$$

where,

$$\xi(h) = \operatorname{atanh}(\tanh(\beta)\tanh(h)).$$

TU/e Technische Universiteit Eindhoven University of Technology

Magnetization



The *magnatization* equals

$$M(\beta, B) = \mathbb{E}\left[\tanh\left(B + \sum_{i=1}^{D} \xi(h_i)\right)\right]$$
$$\approx B + \mathbb{E}[D]\mathbb{E}[\xi(h)].$$

Hence, same scaling for $M(\beta, B)$ and $\mathbb{E}[\xi(h)]$.

Sketch of proof

Taylor expansion of $\mathbb{E}[\xi(h)]$:

$$\begin{aligned}\mathbb{E}[\xi(h)] &= \mathbb{E}\left[\xi\left(B+\sum_{i=1}^{K}\xi(h_i)\right)\right]\\ &\approx \tanh(\beta)\mathbb{E}[h] - C\mathbb{E}[h^3]\\ &= \tanh(\beta)\left(B+\mathbb{E}[K]\mathbb{E}[\xi(h)]\right) - C\mathbb{E}\left[\left(B+\sum_{i=1}^{K}\xi(h_i)\right)^3\right].\end{aligned}$$

Sketch of proof

Taylor expansion of $\mathbb{E}[\xi(h)]$:

$$\begin{aligned}
\mathbb{E}[\xi(h)] &= \mathbb{E}\left[\xi\left(B + \sum_{i=1}^{K} \xi(h_i)\right)\right] \\
&\approx \tanh(\beta)\mathbb{E}[h] - C\mathbb{E}[h^3] \\
&= \tanh(\beta)\left(B + \mathbb{E}[K]\mathbb{E}[\xi(h)]\right) - C\mathbb{E}\left[\left(B + \sum_{i=1}^{K} \xi(h_i)\right)^3\right].
\end{aligned}$$

Only allowed for $\mathbb{E}[K^3] < \infty$. In that case

$$\mathbb{E}[\xi(h)] \approx \tanh(\beta)B + \tanh(\beta)\mathbb{E}[K]\mathbb{E}[\xi(h)] - C\mathbb{E}[\xi(h)]^3.$$

Critical exponent $\boldsymbol{\beta}$

For $\mathbb{E}[K^3] < \infty$,

$$\mathbb{E}[\xi(h)] \approx \tanh(\beta)B + \tanh(\beta)\mathbb{E}[K]\mathbb{E}[\xi(h)] - C\mathbb{E}[\xi(h)]^3.$$

For $\beta > \beta_c$ and $B \searrow 0$,

$$1 \approx \tanh(\beta)\mathbb{E}[K] - C\mathbb{E}[\xi(h_0)]^2.$$

Hence,

$$\mathbb{E}[\xi(h_0)] \approx \left(\frac{\tanh(\beta)\mathbb{E}[K] - 1}{C}\right)^{1/2} \asymp (\beta - \beta_c)^{1/2},$$

thus

$$\boldsymbol{\beta} = 1/2.$$

Critical exponent δ

For $\mathbb{E}[K^3] < \infty$,

$$\mathbb{E}[\xi(h)] \approx \tanh(\beta)B + \tanh(\beta)\mathbb{E}[K]\mathbb{E}[\xi(h)] - C\mathbb{E}[\xi(h)]^3.$$

For $\beta = \beta_c$ and $B > 0$,

$$\mathbb{E}[\xi(h_c)] \approx \tanh(\beta_c)B + 1\mathbb{E}[\xi(h_c)] - C\mathbb{E}[\xi(h_c)]^3.$$

Hence,

$$\mathbb{E}[\xi(h_c)] \approx \left(\frac{\tanh(\beta_c)B}{C}\right)^{1/3} \asymp B^{1/3},$$

thus

$$\delta = 3.$$

The case $\tau \in (3, 5)$

For $\tau \in (3, 5)$, write

$$\mathbb{E}[\xi(h)] = \tanh(\beta)\,(B + \mathbb{E}[K]\mathbb{E}[\xi(h)]) \\ + \mathbb{E}\left[\xi\left(B + \sum_{i=1}^{K} \xi(h_i)\right) - \tanh(\beta)\,(B + K\mathbb{E}[\xi(h)])\right].$$

By taking degrees into account precisely, we can show that

$$\mathbb{E}[\xi(h)] \approx \tanh(\beta)\,(B + \mathbb{E}[K]\mathbb{E}[\xi(h)]) - C\mathbb{E}[\xi(h)]^{\tau-2},$$

yielding

$$\boldsymbol{\beta} = 1/(\tau - 3) \qquad \text{and} \qquad \boldsymbol{\delta} = \tau - 2.$$

Conclusion

Theorem (DGvdH, '12)

	$\mathbb{E}[K^3] < \infty$	$\tau \in (3,4) \cup (4,5)$
β	$1/2$	$1/(\tau - 3)$
δ	3	$\tau - 2$
γ	1	1

Technische Universiteit Eindhoven University of Technology

Conclusion

Theorem (DGvdH, '12)

	$\mathbb{E}[K^3] < \infty$	$\tau \in (3,4) \cup (4,5)$
β	$1/2$	$1/(\tau - 3)$
δ	3	$\tau - 2$
γ	1	1

Conjectured that also

	$\mathbb{E}[K^3] < \infty$	$\tau \in (3,4) \cup (4,5)$
γ'	1	1
α'	0	$(\tau - 5)/(\tau - 3)$