Critical behavior of the inhomogeneous Curie-Weiss model

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Joint work with

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Overview

1. Inhomogeneous Curie-Weiss model (ICW)
2. Motivation via random graphs
3. Critical exponents
4. Non-classical limit theorem

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Inhomogeneous Curie-Weiss model

Fix a sequence of positive weights $(w_i)_{i\geq 1}$

Let ℓ_n be the sum of the first n weights: $\ell_n = \sum_{i=1}^n w_i$

Let σ be a spin configuration $\sigma = (\sigma_1, \ldots, \sigma_n) \in \{-1, +1\}^n$

Define the *Hamiltonian*

$$-H_n(\sigma) = \frac{\beta}{2\ell_n} \sum_{i,j=1}^n w_i w_j \sigma_i \sigma_j + h \sum_{i=1}^n \sigma_i = \frac{\beta}{2\ell_n} \left(\sum_{i=1}^n w_i \sigma_i \right)^2 + h \sum_{i=1}^n \sigma_i$$

where $\beta \geq 0$ is *inverse temperature* and h the *external field*

Inhomogeneous Curie-Weiss model (cont.)

RUB

For Hamiltonian

$$-H_n(\sigma) = \frac{\beta}{2\ell_n}\left(\sum_{i=1}^{n} w_i\sigma_i\right)^2 + h\sum_{i=1}^{n}\sigma_i$$

define Gibbs measure of ICW as

$$\mu_n(\sigma) = \frac{1}{Z_n}e^{-H_n(\sigma)}$$

where Z_n is the normalization *(partition function)*

$$Z_n = \sum_{\sigma\in\{-1,+1\}^n} e^{-H_n(\sigma)}$$

Note that this is the standard Curie-Weiss model if $w \equiv 1$

Assumptions on weights

Let $V_n \sim Uniform\{1, \ldots, n\}$ and $W_n = w_{V_n}$

Assume there exists a random variable W such that, as $n \to \infty$,

(i) $W_n \xrightarrow{\mathcal{D}} W$

(ii) $\mathbb{E}[W_n^2] = \frac{1}{n} \sum_{i=1}^{n} w_i^2 \to \mathbb{E}[W^2] < \infty$

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Generalized random graphs

The *generalized random graph* is defined as follows

Assign to each vertex $i \in \{1, \dots, n\}$ a weight w_i

For each pair of vertices i, j draw an edge between them with probability

$$p_{i,j} = \frac{w_i w_j}{\ell_n + w_i w_j} \quad \left(\approx \frac{w_i w_j}{\ell_n}\right)$$

independently of everything else

Denote the resulting edge set by E_n and
the expectation of such random graphs by Q_n

Annealed Ising model on GRGs

Annealed Gibbs measure

$$\nu_n(\sigma) = \frac{Q_n\left(\exp\left\{\beta \sum_{(i,j)\in E_n} \sigma_i\sigma_j + h\sum_{i=1}^n \sigma_i\right\}\right)}{Q_n(Z_{G_n})}$$

$$= \frac{Q_n\left(\exp\left\{\beta \sum_{i<j} \mathbb{1}_{\{(i,j)\in E_n\}}\sigma_i\sigma_j + h\sum_{i=1}^n \sigma_i\right\}\right)}{Q_n(Z_{G_n})}$$

Because of independence of edges expectation factorizes

Hence we should compute

$$Q_n\left(e^{\beta \mathbb{1}_{\{(i,j)\in E_n\}}\sigma_i\sigma_j}\right)$$

Annealed Ising model on GRGs

$$Q_n\left(e^{\beta \mathbb{1}_{\{(i,j)\in E_n\}}\sigma_i\sigma_j}\right) = p_{i,j}e^{\beta\sigma_i\sigma_j} + (1-p_{i,j})\cdot 1 = e^{\log\left(1+p_{i,j}(e^{\beta\sigma_i\sigma_j}-1)\right)}$$

Using that $\sigma_i\sigma_j$ can only take values -1 and $+1$ and writing

$$\mathbb{1}_{\{\sigma_i\sigma_j=\pm 1\}} = \frac{1}{2}(1\pm\sigma_i\sigma_j)$$

we can write

$$\begin{aligned}
Q_n\left(e^{\beta \mathbb{1}_{\{(i,j)\in E_n\}}\sigma_i\sigma_j}\right) &= e^{\log\left(1+p_{i,j}(e^{\beta}-1)\right)\frac{1}{2}(1+\sigma_i\sigma_j)+\log\left(1+p_{i,j}(e^{-\beta}-1)\right)\frac{1}{2}(1-\sigma_i\sigma_j)} \\
&= C_{i,j}e^{\frac{1}{2}\left(\log\left(1+p_{i,j}(e^{\beta}-1)\right)-\log\left(1+p_{i,j}(e^{-\beta}-1)\right)\right)\sigma_i\sigma_j} \\
&= C_{i,j}e^{\left(p_{i,j}\frac{e^{\beta}-e^{-\beta}}{2}+\mathcal{O}(p_{i,j}^2)\right)\sigma_i\sigma_j} \approx C_{i,j}e^{\frac{w_iw_j}{\ell_n}\sinh\beta\,\sigma_i\sigma_j}
\end{aligned}$$

Annealed Ising model on GRGs

RUB

Using

$$Q_n\left(e^{\beta \mathbb{1}_{\{(i,j)\in E_n\}}\sigma_i\sigma_j}\right) \approx C_{i,j} e^{\sinh \beta \frac{w_i w_j}{\ell_n}\sigma_i\sigma_j}$$

we get

$$\nu_n(\sigma) = \frac{Q_n\left(exp\left\{\beta \sum_{i<j} \mathbb{1}_{\{(i,j)\in E_n\}}\sigma_i\sigma_j + h\sum_{i=1}^n \sigma_i\right\}\right)}{Q_n(Z_{G_n})}$$

$$\approx \frac{\exp\left\{\frac{\sinh \beta}{2\ell_n}\sum_{i,j=1}^n w_i w_j \sigma_i\sigma_j + h\sum_{i=1}^n \sigma_i\right\}}{Z_n}$$

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$$\nu_n(\sigma) = \frac{Q_n\left(exp\left\{\beta\sum_{i<j}\mathbb{1}_{\{(i,j)\in E_n\}}\sigma_i\sigma_j + h\sum_{i=1}^n\sigma_i\right\}\right)}{Q_n(Z_{G_n})}$$

$$\approx \frac{\exp\left\{\frac{\sinh\beta}{2\ell_n}\sum_{i,j=1}^n w_i w_j\sigma_i\sigma_j + h\sum_{i=1}^n\sigma_i\right\}}{Z_n}$$

The same as $\mu_n(\sigma)$ with β replaced by $\sinh\beta$

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Solution of the model

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Theorem (GGvdHP, '16)

For all $\beta \geq 0$ *and* $h \in \mathbb{R}$ *the* pressure *equals*

$$\lim_{n\to\infty} \frac{1}{n} \log Z_n = \log 2 - \sup_z \left\{ \frac{z^2}{2} - \mathbb{E}\left[\log\cosh\left(\sqrt{\frac{\beta}{\mathbb{E}[W]}} Wz + h \right) \right] \right\}$$

Proof uses Hubbard-Stratonovich transform and large deviations of normal random variables

The optimizer satisfies

$$z^\star = \mathbb{E}\left[\tanh\left(\sqrt{\frac{\beta}{\mathbb{E}[W]}} Wz^\star + h \right) \sqrt{\frac{\beta}{\mathbb{E}[W]}} W \right]$$

If $h \neq 0$ this has unique solution with same sign as h

Magnetization

Define the *magnetization* as

$$M_n(\beta, B) := \mu_n\left(\frac{1}{n}\sum_{i=1}^n \sigma_i\right)$$

$$= \sum_{\sigma}\left(\frac{1}{n}\sum_{i=1}^n \sigma_i\right)\frac{e^{\frac{\beta}{2\ell_n}\left(\sum_{i=1}^n w_i\sigma_i\right)^2 + h\sum_{i=1}^n \sigma_i}}{Z_n} = \frac{\partial}{\partial h}\frac{1}{n}\log Z_n$$

Hence one can expect that

$$\lim_{n\to\infty} M_n(\beta, h) = \frac{\partial}{\partial h}\lim_{n\to\infty}\frac{1}{n}\log Z_n = \mathbb{E}\left[\tanh\left(\sqrt{\frac{\beta}{\mathbb{E}[W]}}Wz^\star + h\right)\right]$$

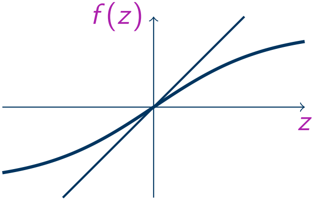
This is indeed true for $h \neq 0$ (GGvdHP, '16)

Critical temperature

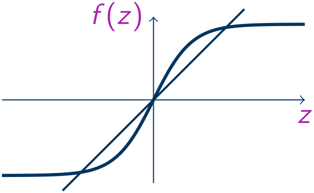
To determine critical temperature take limit $h \to 0$ of fixed point equation

$$z_0^\star = \mathbb{E}\left[\tanh\left(\sqrt{\frac{\beta}{\mathbb{E}[W]}}Wz_0^\star\right)\sqrt{\frac{\beta}{\mathbb{E}[W]}}W\right] =: f(z_0^\star)$$

$f(z)$ is increasing, bounded and concave for positive z



$f'(0) < 1$



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We can compute

$$f'(0) = \beta\frac{\mathbb{E}[W^2]}{\mathbb{E}[W]}$$

and hence

$$\beta_c = \frac{\mathbb{E}[W]}{\mathbb{E}[W^2]}$$

For $\beta > \beta_c$

$$M(\beta, 0^+) := \lim_{h \searrow 0} \lim_{n\to\infty} M_n(\beta, h) > 0.$$

Critical exponents

The critical exponents for the magnetization are defined as

$$M(\beta, 0^+) \asymp (\beta - \beta_c)^{\beta} \qquad \text{for } \beta \searrow \beta_c$$
$$M(\beta_c, h) \asymp h^{1/\delta} \qquad \text{for } h \searrow 0$$

Furthermore, we can define the *susceptibility* as

$$\chi(\beta, h) = \frac{\partial}{\partial h} M(\beta, h)$$

Its critical exponents are

$$\chi(\beta, 0^+) \asymp (\beta_c - \beta)^{-\gamma} \qquad \text{for } \beta \nearrow \beta_c$$
$$\chi(\beta, 0^+) \asymp (\beta - \beta_c)^{-\gamma'} \qquad \text{for } \beta \searrow \beta_c$$

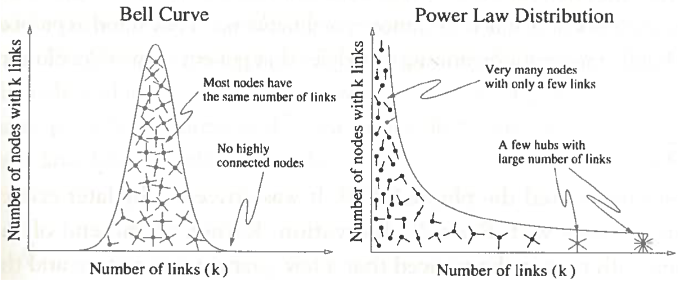
Additional weight assumptions

We distinguish two cases

(i) $\mathbb{E}[W^4] < \infty$

(ii) W obeys a *power law* with *exponent* $\tau \in (3,5]$, i.e., there exist constants $C_W > c_W > 0$ and $w_0 > 0$ such that

$$c_W w^{-(\tau-1)} \leq \mathbb{P}[W > w] \leq C_W w^{-(\tau-1)} \qquad \forall w > w_0$$

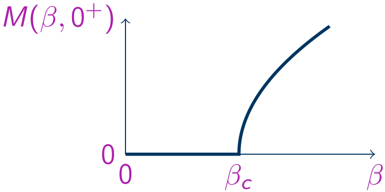


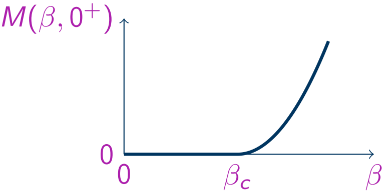
Barabási, Linked, '02

Critical exponents for ICW

Theorem (DGGvdHP, '15)

	$\mathbb{E}[W^4] < \infty$	$\tau \in (3,5)$
$\boldsymbol{\beta}$	$1/2$	$1/(\tau-3)$
$\boldsymbol{\delta}$	3	$\tau-2$
γ, γ'	1	1





Critical exponents for ICW

RUB

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$\boldsymbol{\delta}$	3	$\tau-2$
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For $\tau = 5$ we have the following logarithmic corrections

$$M(\beta, 0^+) \asymp \left(\frac{\beta-\beta_c}{\log 1/(\beta-\beta_c)}\right)^{1/2} \qquad M(\beta_c, h) \asymp \left(\frac{h}{\log(1/h)}\right)^{1/3}$$

Mean-field exponents

RUB

The values

$$\beta = 1/2 \qquad \delta = 3 \qquad \gamma, \gamma' = 1$$

are called *mean-field* values. They are the same for

- Curie-Weiss model
- Ising model on $\mathbb{Z}^d, d > 4$
  (β, δ : Aizenman, Fernández, '86, γ : Aizenman, '82)
- Many other models

Note that these values do *not* hold for $\tau \leq 5$

Despite the fact that this is still a mean-field model!

Sketch of proof

Note that

$$M(\beta, h) = \mathbb{E}\left[\tanh\left(\sqrt{\frac{\beta}{\mathbb{E}[W]}} W z^\star + h\right)\right] \approx \sqrt{\beta \mathbb{E}[W]} z^\star + h.$$

Hence, same scaling for $M(\beta, h)$ and $z^\star$

Using Taylor expansion $\tanh(x) \approx x - \frac{1}{3}x^3$

$$\begin{aligned} z^\star &= \mathbb{E}\left[\tanh\left(\sqrt{\frac{\beta}{\mathbb{E}[W]}} W z^\star + h\right) \sqrt{\frac{\beta}{\mathbb{E}[W]}} W\right] \\ &\approx \sqrt{\beta \mathbb{E}[W]} h + \beta \frac{\mathbb{E}[W^2]}{\mathbb{E}[W]} z^\star - \frac{\beta^2}{3} \frac{\mathbb{E}[W^4]}{\mathbb{E}[W]^2} z^{\star 3} \end{aligned}$$

Sketch of proof

Note that

$$M(\beta,h)=\mathbb{E}\left[\tanh\left(\sqrt{\frac{\beta}{\mathbb{E}[W]}}Wz^\star+h\right)\right]\approx\sqrt{\beta\mathbb{E}[W]}z^\star+h.$$

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Only allowed for $\mathbb{E}[W^4]<\infty$ *!*

Critical exponent β

For $\mathbb{E}[W^4] < \infty$

$$z^\star \approx \sqrt{\beta \mathbb{E}[W]} h + \beta \frac{\mathbb{E}[W^2]}{\mathbb{E}[W]} z^\star - \frac{\beta^2}{3} \frac{\mathbb{E}[W^4]}{\mathbb{E}[W]^2} z^{\star 3}$$

For $\beta > \beta_c$ and $h \searrow 0$

$$1 \approx \beta \frac{\mathbb{E}[W^2]}{\mathbb{E}[W]} - \frac{\beta^2}{3} \frac{\mathbb{E}[W^4]}{\mathbb{E}[W]^2} z^{\star 2}$$

Hence

$$z^\star \approx \left(\frac{\beta \frac{\mathbb{E}[W^2]}{\mathbb{E}[W]} - 1}{C} \right)^{\frac{1}{2}} = \left(\frac{\beta - \beta_c}{\beta_c C} \right)^{\frac{1}{2}}$$

thus

$$\boldsymbol{\beta} = 1/2$$

Critical exponent δ

For $\mathbb{E}[W^4] < \infty$

$$z^\star \approx \sqrt{\beta \mathbb{E}[W] h} + \beta \frac{\mathbb{E}[W^2]}{\mathbb{E}[W]} z^\star - \frac{\beta^2}{3} \frac{\mathbb{E}[W^4]}{\mathbb{E}[W]^2} z^{\star 3}$$

For $\beta = \beta_c$ and $h > 0$

$$z^\star \approx \sqrt{\beta_c \mathbb{E}[W] h} + 1 \cdot z^\star - \frac{\beta_c^2}{3} \frac{\mathbb{E}[W^4]}{\mathbb{E}[W]^2} z^{\star 3}$$

Hence

$$z^\star \approx C\, h^{1/3}$$

thus

$$\boldsymbol{\delta} = 3$$

The case $\tau \in (3, 5)$

For $\tau \in (3, 5)$ write

$$z^\star = \sqrt{\beta \mathbb{E}[W]} h + \beta \frac{\mathbb{E}[W^2]}{\mathbb{E}[W]} z^\star + \mathbb{E}\left[\left\{ \tanh\left(\sqrt{\frac{\beta}{\mathbb{E}[W]}} W z^\star + h \right) - \left(h + \sqrt{\frac{\beta}{\mathbb{E}[W]}} W z^\star \right) \right\} \sqrt{\frac{\beta}{\mathbb{E}[W]}} W \right]$$

By splitting analysis for *small* and *large* W we can show that

$$z^\star \approx \sqrt{\beta \mathbb{E}[W]} h + \beta \frac{\mathbb{E}[W^2]}{\mathbb{E}[W]} z^\star - C {z^\star}^{\tau - 2}$$

yielding

$$\boldsymbol{\beta} = 1/(\tau - 3) \qquad \text{and} \qquad \boldsymbol{\delta} = \tau - 2$$

Truncated moments

For $a > \tau - 1$

$$\mathbb{E}[W^a \mathbb{1}_{\{W \leq \ell\}}] \sim \int_1^\ell w^a w^{-\tau} \mathrm{d}w \sim \ell^{a-(\tau-1)}$$

Similarly, for $a < \tau - 1$

$$\mathbb{E}[W^a \mathbb{1}_{\{W > \ell\}}] \sim \int_\ell^\infty w^a w^{-\tau} \mathrm{d}w \sim \ell^{a-(\tau-1)}$$

Optimal choice is $\ell = 1/z^\star$

Use of truncated moments

For *small* W use Taylor expansion as before

Then estimate terms like

$$\mathbb{E}[W^4 \mathbb{1}_{\{W \leq 1/z^\star\}}] z^{\star 3} \sim z^{\star -(4-(\tau-1))} z^{\star 3} = z^{\star \tau - 2}$$

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For *large* W use simpler bounds e.g. a constant

Then estimate terms like

$$\mathbb{E}[W^2 \mathbb{1}_{\{W > 1/z^\star\}}] z^{\star} \sim z^{\star -(2-(\tau-1))} z^{\star} = z^{\star \tau - 2}$$

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Central limit theorem

Let S_n be the total spin $S_n = \sum_{i=1}^{n} \sigma_i$

Theorem (GGvdHP, '16)

For $\beta \geq 0, h \neq 0$ *and for* $0 \leq \beta < \beta_c, h = 0$

$$\frac{S_n - \mu_n(S_n)}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \chi)$$

Proved by analyzing *cumulant generating function*

$$c_n(t) = \frac{1}{n} \log \mu_n\left(e^{tS_n}\right)$$

and its first two derivatives

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and its first two derivatives

Does *not* hold for $\beta = \beta_c, h = 0$

Additional weight assumptions

We distinguish two cases

(i) $\mathbb{E}[W_n^4] = \frac{1}{n}\sum_{i\in[n]} w_i^4 \to \mathbb{E}[W^4] < \infty$

(ii) the weights are chosen according the deterministic sequence

$$w_i = c_w \left(\frac{n}{i}\right)^{1/(\tau-1)}$$

for some constant $c_w > 0$ and $\tau \in (3,5)$

The latter is a stronger assumption on the power-law behavior of $(w_i)_{i\geq 1}$

Non-classical limit theorem

Theorem (DGGvdHP, '15)

Let $\beta = \beta_c, h = 0$. Then, there exists a random variable X such that

$$\frac{S_n}{n^{\delta/(\delta+1)}} \xrightarrow{\mathcal{D}} X$$

where X has density proportional to $exp(-f(x))$ with

$$f(x) = \begin{cases} \frac{1}{12}\frac{\mathbb{E}[W^4]}{\mathbb{E}[W]^4}x^4 & \text{when } \mathbb{E}[W^4] < \infty \\ \sum_{i\geq 1}\left(\frac{1}{2}\left(\frac{\tau-2}{\tau-1}\, x\, i^{-1/(\tau-1)}\right)^2 \right. & \left. - \log\cosh\left(\frac{\tau-2}{\tau-1}\, x\, i^{-1/(\tau-1)}\right)\right) \\ & \text{when } \tau \in (3,5) \end{cases}$$

In both cases there exists an explicit constant $C > 0$ such that

$$\lim_{x\to\infty} \frac{f(x)}{x^{1+\delta}} = C$$

Sketch of proof

It suffices to show that exponential moments converge

$$\lim_{n\to\infty} \mu_n \left(\exp\left\{ r \frac{S_n}{n^{\delta/(\delta+1)}} \right\} \right) \to \frac{\int_{-\infty}^{\infty} \exp(rx - f(x))\mathrm{d}x}{\int_{-\infty}^{\infty} \exp(-f(x))\mathrm{d}x}$$

Use Hubbard-Stratonovich transform ($e^{t^2/2} = \mathbb{E}[e^{tZ}]$ with $Z \sim \mathcal{N}(0,1)$) to write

$$\sum_{\sigma} e^{\frac{r}{n^{\delta/(\delta+1)}} \sum_{i=1}^n \sigma_i} e^{\frac{\beta}{2\ell_n} \left(\sum_{i=1}^n w_i \sigma_i\right)^2} = C_n \int_{-\infty}^{\infty} e^{-nG_n\left(\frac{x}{n^{1/(\delta+1)}}, r\right)} \mathrm{d}x$$

with
$$G_n(x,r) = \frac{x^2}{2} - \mathbb{E}\left[\log\cosh\left(\sqrt{\frac{\beta}{\mathbb{E}[W]}} Wx + \frac{r}{n^{\delta/(\delta+1)}} \right) \right]$$

The result then follows by using the Taylor expansion (for $\mathbb{E}[W^4] < \infty$)

$$\log\cosh x \approx \frac{x^2}{2} - \frac{1}{12}x^4$$

Related / future research

Results on critical behavior (DGvdH14) and CLTs (GGvdHP15) also hold in *quenched* model

What about non-classical limit theorem in quenched model?

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What about non-classical limit theorem in quenched model?

What about *rates of convergence* for limit theorems?

Perhaps we can use Stein's method for exchangeable pairs

Successfully used for Curie-Weiss model in Eichelsbacher, Löwe 2010

Related / future research

Results on critical behavior (DGvdH14) and CLTs (GGvdHP15) also hold in *quenched* model

What about non-classical limit theorem in quenched model?

What about *rates of convergence* for limit theorems?

Perhaps we can use Stein's method for exchangeable pairs

Successfully used for Curie-Weiss model in Eichelsbacher, Löwe 2010

The solution of the model and in several cases the critical behavior can also be derived for *compact spins*, e.g., continuous spins on $[-1, 1]$
D., Külske, Schriever 2016