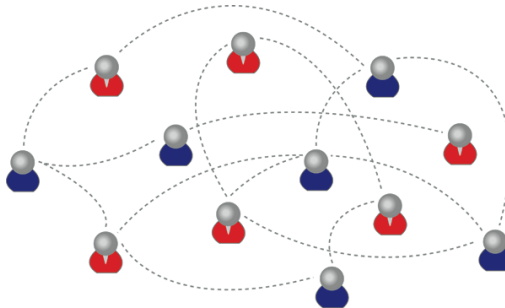# Metastability of the Ising model on random regular graphs at zero temperature

Sander Dommers

Ising model on random graphs

We model opinion spreading with the *Ising model,* a paradigm model in statistical physics for *cooperative behavior.*

What are effects of *structure* of the network on *behavior* of Ising model?

Random regular graphs

Construction of *random r-regular graph* with n vertices:

- *Assign r* half-edges to each vertex $i \in \{1, \ldots, n\}$

- *Attach* first half-edge to another half-edge *uniformly at random*

- Continue until all half-edges are connected

Denote resulting graph by $G_n = (\{1, \ldots, n\}, E_n)$

Ising model

Ising measure on G_n for $\sigma \in \{-1, +1\}^n$

$$\mu(\sigma) = \frac{1}{Z_n} e^{-\beta H(\sigma)}$$

with Hamiltonian

$$H(\sigma) = -\sum_{(i,j) \in E_n} \sigma_i \sigma_j - h \sum_{i=1}^{n} \sigma_i$$

where

$\beta \geq 0$ inverse temperature

$h > 0$ external magnetic field

Z_n normalization factor (partition function)

Discrete Glauber dynamics

Discrete time *Markov chain* where at every time step

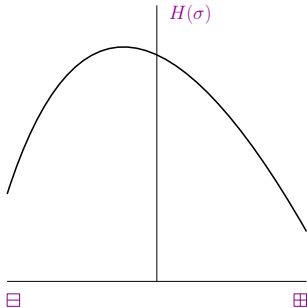1) Pick vertex i uniformly from $\{1, \ldots, n\}$

2) Flip spin σ_i with probability $\begin{cases} 1 & \text{if } H(\sigma^i) \leq H(\sigma) \\ \\ e^{-\beta[H(\sigma^i) - H(\sigma)]} & \text{if } H(\sigma^i) > H(\sigma) \end{cases}$

Equilibrium distribution $\mu(\sigma) = \frac{1}{Z_n} e^{-\beta H(\sigma)}$

Metastability

Take zero temperature limit $\beta \to \infty$

Then, $\mu(\sigma) = \frac{1}{Z_n} e^{-\beta H(\sigma)}$ concentrates on minimizer of

$$H(\sigma) = -\sum_{(i,j) \in E_n} \sigma_i \sigma_j - h \sum_{i=1}^{n} \sigma_i$$

Minimizer is ⊞ → *stable* state

If h small, local minimum ⊟
→ *metastable* state

Time $\tau(\boxminus, \boxplus)$ to go from ⊟ to ⊞?

Main result

Theorem (D. 2015)

For random r-regular graphs with $r \geq 3$, h small, whp, there exist constants $0 < C_1 < \infty, C_2 < \infty$ such that

$$\lim_{\beta \to \infty} \mathbb{P}[\exp\{\beta(r/2 - C_1\sqrt{r})n\} < \tau(\boxminus, \boxplus) < \exp\{\beta(r/2 + C_2\sqrt{r})n\}] = 1$$
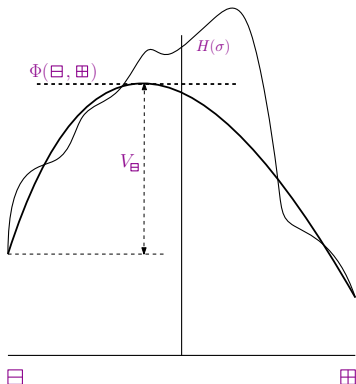
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Note that exponent is *linear* in n

Big difference with e.g. lattices where $\tau(\boxminus, \boxplus) \sim e^{\beta C}$

Pathwise approach

Cassandro, Galves, Olivieri, Vares, 1984, Neves, Schonmann, 1991
Manzo, Nardi, Olivieri, Scoppola, 2004, Cirillo, Nardi 2013

Communication height

$$\Phi(\sigma, \sigma') = \min_{\omega \text{ path from } \sigma \text{ to } \sigma'} \max_{\sigma'' \in \omega} H(\sigma'')$$

Stability level

$$V_\sigma = \min_{\sigma': H(\sigma') < H(\sigma)} \Phi(\sigma, \sigma') - H(\sigma)$$

Pathwise approach (cont.)

Proposition

If there exist $0 < \Gamma_\ell \leq \Gamma_u < \infty$ such that

    1) $\Phi(\boxminus, \boxplus) - H(\boxminus) \geq \Gamma_\ell$

    2) $\Phi(\boxminus, \boxplus) - H(\boxminus) \leq \Gamma_u$

    3) for all $\sigma \notin \{\boxminus, \boxplus\}$ *it holds that* $V_\sigma \leq \Gamma_u$

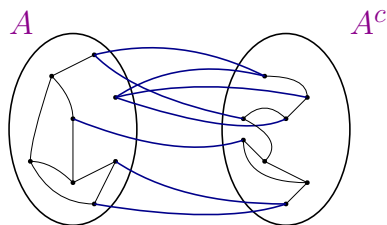*then, for all* $\varepsilon > 0$,

$$\lim_{\beta \to \infty} \mathbb{P}[\exp\{\beta(\Gamma_\ell - \varepsilon)\} < \tau(\boxminus, \boxplus) < \exp\{\beta(\Gamma_u + \varepsilon)\}] = 1$$
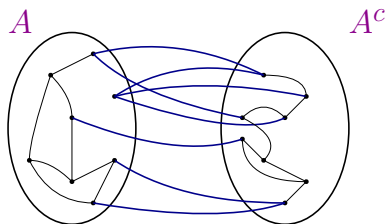
Isoperimetric number

(Edge) isoperimetric number

$$i_e(G_N) = \min_{\substack{A \subset \{1,\dots,n\} \\ |A| \leq n/2}} \frac{|\partial_e A|}{|A|}$$

$$(|\partial_e A| \geq i_e(G_N)|A|)$$

Isoperimetric number

A $\qquad$ A^c

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For r-regular random graphs $\exists\, C > 0$ such that, whp,
(Bollobás, 1988, Alon, 1997)

$$\frac{r}{2} - \sqrt{\log 2}\sqrt{r} \leq i_e(G_n) \leq \frac{r}{2} - C\sqrt{r}$$

(For lattices $i_e \to 0$ as $n \to \infty$)

Lower bound

Lemma

If h small, then $\exists\, C_1 > 0$ such that $\forall\, \sigma$ with $n/2 +$ spins

$$H(\sigma) - H(\boxminus) \geq (i_e(G_n) - h)n \geq \left(\frac{r}{2} - C_1\sqrt{r}\right) n$$

Lower bound

Lemma
If h small, then $\exists\, C_1 > 0$ such that $\forall\, \sigma$ with $n/2$ + spins

$$H(\sigma) - H(\boxminus) \geq (i_e(G_n) - h)n \geq \left(\frac{r}{2} - C_1\sqrt{r}\right)n$$

Proof.

$$H(\boxminus) = -\sum_{(i,j)\in E_n}(-1)^2 - h\sum_{i=1}^{n}(-1) = -|E_n| + h\,n$$
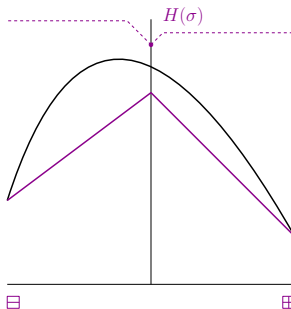
With A set of + spins

$$H(\sigma) = -|E_n| + 2|\partial_e A| - \frac{n}{2}h + \frac{n}{2}h \geq 2i_e(G_n)\frac{n}{2} - |E_n|$$

Upper bound

For the upper bound use that there exists a configuration with $n/2$ with

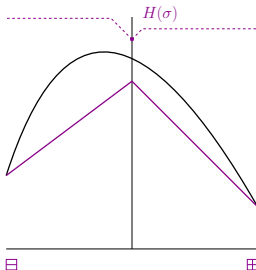$$|\partial_e A| \leq \left(\frac{r}{2} - C\sqrt{r}\right)\frac{n}{2}$$

and that there always exists a path to ⊟ and ⊞ that doesn't increase energy too much

Main result

Theorem (D. 2015)

For random r-regular graphs with $r \geq 3$, h small, whp, there exist constants $0 < C_1 < \infty, C_2 < \infty$ such that

$$\lim_{\beta \to \infty} \mathbb{P}[\exp\{\beta(r/2 - C_1\sqrt{r})n\} < \tau(\boxminus, \boxplus) < \exp\{\beta(r/2 + C_2\sqrt{r})n\}] = 1$$

Extension to more general degrees

Theorem (D., den Hollander, Jovanovski, Nardi, 2016)

Let all degrees be at least 3. *Then, whp,* $\exists\, 0 < \gamma_\ell \leq \gamma_u < \infty$ *such that*

$$\lim_{\beta \to \infty} \mathbb{P}[\exp\{\beta\,\gamma_\ell\,n\} < \tau(\boxminus, \boxplus) < \exp\{\beta\,\gamma_u\,n\}] = 1$$

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Open problem

Prove that $\tau(\boxminus, \boxplus) \sim \exp\{\beta\,\Gamma_n\}$ with

$$\lim_{n \to \infty} \frac{\Gamma_n}{n} = \gamma$$

and determine γ