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Continuous spin models on annealed generalized random graphs

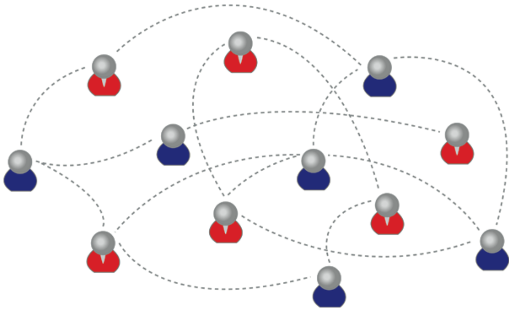
Sander Dommers

Joint work with

Christof Külske and Philipp Schriever

Motivation

Random graphs can model complex networks, e.g., social networks



Spin models can model for example opinion formation

Overview

1. Spin models on annealed generalized random graphs
2. Pressure
3. Mean-field equation
4. Critical behavior of systems with 2$^{\mathrm{nd}}$ order phase transition

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Generalized random graphs

The *generalized random graph* is defined as follows

Assign to each vertex $i \in \{1, \ldots, n\}$ a weight w_i

Let ℓ_n be the sum of the weights: $\ell_n = \sum_{i=1}^n w_i$

For each pair of vertices i, j draw an edge between them with probability

$$p_{i,j} = \frac{w_i w_j}{\ell_n + w_i w_j} \quad \left(\approx \frac{w_i w_j}{\ell_n} \right)$$

independently of everything else

Denote the resulting edge set by E_n and
the expectation of such random graphs by Q_n^w

Assumptions on weights

Let $V_n \sim Uniform\{1, \ldots, n\}$ and $W_n = w_{V_n}$

Assume there exists a random variable W with distribution P such that, as $n \to \infty$,

(i) $W_n \xrightarrow{\mathcal{D}} W$

(ii) $\mathbb{E}[W_n^2] = \frac{1}{n}\sum_{i=1}^n w_i^2 \to \mathbb{E}[W^2] < \infty$

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Consequences:

$$\max_{i=1}^{n} w_i = o(n^{1/2})$$

and

$$p_{i,j} = \frac{w_i w_j}{\ell_n + w_i w_j} \approx \frac{w_i w_j}{\ell_n} \approx \frac{w_i w_j}{n\mathbb{E}[W]}$$

Annealed spin model

Let S be a compact Polish space

Denote a spin configuration by $\sigma = (\sigma_i)_{i \in \{1,\dots,n\}} \in S^n$

Let α be an a priori probability measure on S

Let $\Phi : S \times S \to \mathbb{R}$ be a bounded interaction potential

The *annealed Gibbs measure* is given by

$$\mu_n(\mathrm{d}\sigma) = \frac{Q_n^w \left(e^{\sum_{(i,j)\in E_n} \Phi(\sigma_i,\sigma_j)} \prod_{i=1}^n \alpha(\mathrm{d}\sigma_i) \right)}{Q_n^w\, \alpha^n \left(e^{\sum_{(i,j)\in E_n} \Phi(\tilde{\sigma}_i,\tilde{\sigma}_j)} \right)}$$

Annealed spin model

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Compare *annealed* Gibbs measure

$$\mu_n(\mathrm{d}\sigma) = \frac{Q_n^w \left(e^{\sum_{(i,j)\in E_n} \Phi(\sigma_i,\sigma_j)} \prod_{i=1}^n \alpha(\mathrm{d}\sigma_i) \right)}{Q_n^w \, \alpha^n \left(e^{\sum_{(i,j)\in E_n} \Phi(\tilde\sigma_i,\tilde\sigma_j)} \right)}$$

with *quenched* Gibbs measure

$$\mu_n^{qu}(\mathrm{d}\sigma) = \frac{e^{\sum_{(i,j)\in E_n} \Phi(\sigma_i,\sigma_j)} \prod_{i=1}^n \alpha(\mathrm{d}\sigma_i)}{\alpha^n \left(e^{\sum_{(i,j)\in E_n} \Phi(\tilde\sigma_i,\tilde\sigma_j)} \right)}$$

In *quenched* case: model on *fixed* (random) realization of graph

In *annealed* case: model on *average* realization of graph
Graph changes on much faster timescale as spins

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Annealed pressure

Define the annealed pressure as

$$\psi_n(\Phi, \alpha, w) = \frac{1}{n} \log Q_n^w \, \alpha^n \left(e^{\sum_{(i,j)\in E_n} \Phi(\tilde{\sigma}_i, \tilde{\sigma}_j)} \right)$$

and define the thermodynamic limit of the pressure as

$$\psi(\Phi, \alpha, P) = \lim_{n\to\infty} \psi_n(\Phi, \alpha, w)$$

if this limit exists

Annealing

We can write

$$Q_n^w\left(e^{\sum_{(i,j)\in E_n}\Phi(\sigma_i,\sigma_j)}\right)=Q_n^w\left(e^{\sum_{i<j}\mathbb{1}_{\{(i,j)\in E_n\}}\Phi(\sigma_i,\sigma_j)}\right)$$

Because of *independence* of edges expectation factorizes

We compute

$$\begin{aligned}Q_n^w\left(e^{\mathbb{1}_{\{(i,j)\in E_n\}}\Phi(\sigma_i,\sigma_j)}\right)&=p_{i,j}e^{\Phi(\sigma_i,\sigma_j)}+1-p_{i,j}\\&=e^{\log\left(1+p_{i,j}(e^{\Phi(\sigma_i,\sigma_j)}-1)\right)}\approx e^{p_{i,j}(e^{\Phi(\sigma_i,\sigma_j)}-1)}\\&\approx c_{i,j}e^{\frac{w_iw_j}{n\mathbb{E}[W]}e^{\Phi(\sigma_i,\sigma_j)}}\end{aligned}$$

Annealing

Hence we get

$$Q_n^w\left(e^{\sum_{(i,j)\in E_n}\Phi(\sigma_i,\sigma_j)}\right) \approx Ce^{\sum_{i<j}\frac{w_iw_j}{n\mathbb{E}[W]}e^{\Phi(\sigma_i,\sigma_j)}} \approx Ce^{n\frac{1}{n^2}\sum_{i,j=1}^n\frac{w_iw_je^{\Phi(\sigma_i,\sigma_j)}}{2\mathbb{E}[W]}}$$

Define the empirical distribution

$$L_n^{\sigma,w} = \frac{1}{n}\sum_{i=1}^n \delta_{(\sigma_i,w_i)}$$

and the function

$$U(\sigma,\sigma',w,w') = \frac{ww'e^{\Phi(\sigma,\sigma')}}{2\mathbb{E}[W]}$$

Then

$$Q_n^w\left(e^{\sum_{(i,j)\in E_n}\Phi(\sigma_i,\sigma_j)}\right) \approx Ce^{nL_n^{\sigma,w}\otimes L_n^{\sigma,w}(U)}$$

Large deviations

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We want to compute the pressure

$$\psi(\Phi, \alpha, P) = \lim_{n\to\infty} \frac{1}{n} \log Q_n^w\, \alpha^n \left(e^{\sum_{(i,j)\in E_n} \Phi(\tilde{\sigma}_i, \tilde{\sigma}_j)} \right)$$
$$= \lim_{n\to\infty} \frac{1}{n} \log \alpha^n \left(e^{n L_n^{\sigma,w} \otimes L_n^{\sigma,w}(U)} \right) + C$$

(We ignore the constant from now on)

If we can show that $L_n^{\sigma,w}$ satisfies a *Large Deviations Principle (LDP)* we can use *Varadhan's lemma* to compute the pressure

Variational expression for pressure

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Theorem

$$\psi(\Phi, \alpha, P) = \sup_{\substack{\nu \in \mathcal{M}_1(S, \mathbb{R}_+): \\ \nu(\mathrm{d}w) = P(\mathrm{d}w)}} \nu \otimes \nu(U) - \int S(\nu^w | \alpha) P(\mathrm{d}w)$$

where $S(\nu|\alpha)$ is the relative entropy

$$S(\nu|\alpha) = \int \log \frac{\mathrm{d}\nu}{\mathrm{d}\alpha}(\sigma)\nu(\mathrm{d}\sigma)$$

- Truncate weights to have compact space
- Prove that rate function is ∞ when ν is not a probability measure or $\nu(\mathrm{d}w) \neq P(\mathrm{d}w)$
- Take limit of truncation

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Mean-field equation (heuristics)

Suppose there exists an effective potential $V(\sigma)$ so that

$$\nu^{w}(\mathrm{d}\sigma) = \nu^{w,V}(\mathrm{d}\sigma) := \frac{1}{z} e^{wV(\sigma)} \alpha(\mathrm{d}\sigma)$$

Then

$$\frac{1}{z} e^{wV(\sigma)} \alpha(\mathrm{d}\sigma) \approx \frac{1}{z} e^{w \frac{1}{n} \sum_{j=1}^{n} \frac{w_j}{\mathbb{E}[W]} e^{\Phi(\sigma,\sigma_j)}} \alpha(\mathrm{d}\sigma)$$

so that

$$V(\sigma) \approx \frac{1}{n} \sum_{j=1}^{n} \frac{w_j}{\mathbb{E}[W]} e^{\Phi(\sigma,\sigma_j)} \approx \mathbb{E}\left[\frac{W}{\mathbb{E}[W]} \int e^{\Phi(\sigma,\tilde{\sigma})} \nu^{W,V}(\mathrm{d}\tilde{\sigma}) \right]$$

Mean-field equation

Let $\mathcal{V}$ be the set of solutions to

$$V(\sigma) = \mathbb{E}\left[\frac{W}{\mathbb{E}[W]} \int e^{\Phi(\sigma,\tilde{\sigma})} \nu^{W,V}(\mathrm{d}\tilde{\sigma})\right]$$

Theorem

$$\begin{aligned}\psi(\Phi, \alpha, P) &= \sup_{V \in \mathcal{V}} \mathbb{E}\left[\nu^{W,V} \otimes \nu^{W',V}(U)\right] - \int S(\nu^{w,V}|\alpha) P(\mathrm{d}w) \\ &= \sup_{V \in \mathcal{V}} -\frac{1}{2}\mathbb{E}\left[W\nu^{W,V}(V)\right] + \mathbb{E}\left[\log \alpha(e^{WV})\right]\end{aligned}$$

First equality can be made rigorous using variations

Second equality follows from using fixed point equation

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Rank-2 models on interval

Let $S = [-1, 1]$ and $e^{\Phi(\sigma_i, \sigma_j)} = c + \theta \sigma_i \sigma_j$

Then

$$V(\sigma) = \mathbb{E}\left[\frac{W}{\mathbb{E}[W]} \int e^{\Phi(\sigma,\tilde{\sigma})} \nu^{W,V}(\mathrm{d}\tilde{\sigma})\right] = c + \theta\sigma\mathbb{E}\left[\frac{W}{\mathbb{E}[W]} \int \tilde{\sigma} \nu^{W,V}(\mathrm{d}\tilde{\sigma})\right]$$

Hence $V(\sigma)$ must be of the form $V(\sigma) = c + m\sigma$. Using this

$$\frac{m}{\theta} = \mathbb{E}\left[\frac{W}{\mathbb{E}[W]} \int \tilde{\sigma} \nu^{W,m\tilde{\sigma}}(\mathrm{d}\tilde{\sigma})\right] =: \varphi(m)$$

Cumulant generating function of α

We can write

$$\varphi(m) = \mathbb{E}\left[\frac{W}{\mathbb{E}[W]}\int \sigma \nu^{W,m\sigma}(\mathrm{d}\sigma)\right] = \mathbb{E}\left[\frac{W}{\mathbb{E}[W]}\frac{\int \sigma\, \mathrm{e}^{Wm\sigma}\alpha(\mathrm{d}\sigma)}{\int \mathrm{e}^{Wm\sigma}\alpha(\mathrm{d}\sigma)}\right]$$
$$= \mathbb{E}\left[\frac{W}{\mathbb{E}[W]}\frac{\mathrm{d}}{\mathrm{d}t}\log\alpha(e^{t\sigma})\Big|_{t=Wm}\right]$$

Similarly

$$\varphi''(m) = \mathbb{E}\left[\frac{W^3}{\mathbb{E}[W]}\frac{\mathrm{d}^3}{\mathrm{d}t^3}\log\alpha(e^{t\sigma})\Big|_{t=Wm}\right]$$

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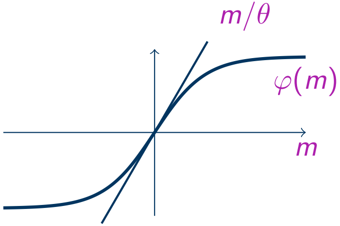
2^{nd} order phase transition

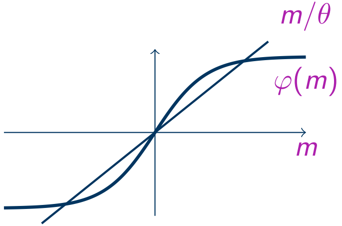
Suppose that α is an even measure with, for all $t > 0$,

$$\frac{\mathrm{d}^3}{\mathrm{d}t^3} \log \alpha(e^{t\sigma}) < 0$$

Then, for $m > 0$, also $\varphi''(m) < 0$ and is hence *concave*



$\varphi'(0) < 1/\theta$



$\varphi'(0) > 1/\theta$

We have a 2^{nd} order phase transition at $\varphi'(0) = 1/\theta_c$

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Critical behavior

Suppose that α is an even measure with, for all $t > 0$,

$$\frac{\mathrm{d}^3}{\mathrm{d}t^3} \log \alpha(e^{t\sigma}) < 0$$

Critical value

$$1/\theta_c = \varphi'(0) = \frac{\mathbb{E}[W^2]}{\mathbb{E}[W]} \alpha(\sigma^2)$$

For $\theta > \theta_c$ unique positive solution m^+

Now suppose that k is the smallest natural number such that

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k} \log \alpha(e^{t\sigma}) \Big|_{t=0} < 0$$

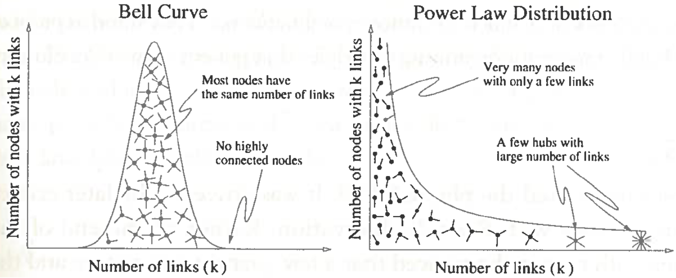
Additional weight assumptions

We distinguish two cases

(i) $\mathbb{E}[W^k] < \infty$

(ii) W obeys a *power law* with *exponent* $\tau \in (3, k+1)$, i.e., there exist constants $C_W > c_W > 0$ and $w_0 > 0$ such that

$$c_W w^{-(\tau-1)} \leq \mathbb{P}[W > w] \leq C_W w^{-(\tau-1)} \qquad \forall w > w_0$$



Barabási, Linked, '02

Critical exponent

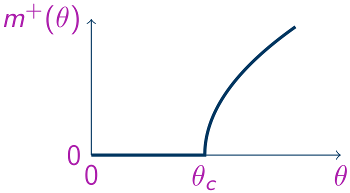
Define the critical exponent β as

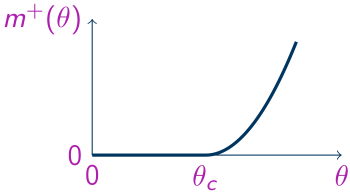
$$m^+(\theta) \asymp (\theta - \theta_c)^{\beta} \qquad \text{for } \theta \searrow \theta_c$$

Theorem

$$\beta = \begin{cases} 1/(k-2) & \text{for } \mathbb{E}[W^k] < \infty \\ 1/(\tau - 3) & \text{for } \tau \in (3, k+1) \end{cases}$$

Example for $k = 4$ and $\mathbb{E}[W^4] < \infty$ (left) and $\tau = 3.5$ (right)





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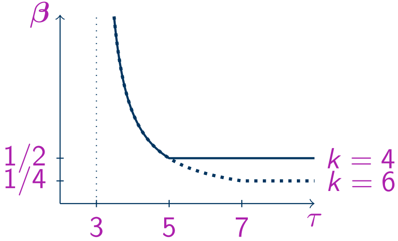

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For $\tau = k+1$ we have the following logarithmic corrections

$$m^+(\theta) \asymp \left(\frac{\theta - \theta_c}{\log 1/(\theta - \theta_c)}\right)^{1/(k-2)}$$

Sketch of proof

Using a Taylor approximation

$$\frac{m^+}{\theta} = \varphi(m^+) \approx \varphi(0) + \varphi'(0)m^+ + \varphi^{(k-1)}(0)\frac{m^{+k-1}}{(k-1)!}$$
$$= \frac{m^+}{\theta_c} + \mathbb{E}\left[\frac{W^k}{\mathbb{E}[W]}\frac{\mathrm{d}^k}{\mathrm{d}t^k}\log\alpha(e^{t\sigma})\Big|_{t=0}\right]\frac{m^{+k-1}}{(k-1)!}$$

Sketch of proof

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Only allowed for $\mathbb{E}[W^k] < \infty$ *!*

Sketch of proof

Using a Taylor approximation

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$$= \frac{m^+}{\theta_c} + \mathbb{E}\left[\frac{W^k}{\mathbb{E}[W]}\frac{\mathrm{d}^k}{\mathrm{d}t^k}\log\alpha(e^{t\sigma})\Big|_{t=0}\right]\frac{m^{+k-1}}{(k-1)!}$$

For $\mathbb{E}[W^k] < \infty$

$$\frac{1}{\theta_c} - \frac{1}{\theta} = \frac{\theta - \theta_c}{\theta\theta_c} \approx -\frac{\mathbb{E}[W^k]}{\mathbb{E}[W]}\frac{\mathrm{d}^k}{\mathrm{d}t^k}\log\alpha(e^{t\sigma})\Big|_{t=0}\frac{m^{+k-2}}{(k-1)!}$$

Indeed

$$m^+ \asymp (\theta - \theta_c)^{1/(k-2)}$$

Sketch of proof

$$\frac{m^+}{\theta} = \varphi(m^+) = \mathbb{E}\left[\frac{W}{\mathbb{E}[W]} \int \sigma \nu^{W,m\sigma}(\mathrm{d}\sigma)\right]$$

For $\tau \in (3, k+1]$ split analysis for small and large values of W

Use properties of truncated moments of W

External magnetic field

Let

$$\alpha_h(\mathrm{d}\sigma) = \frac{1}{z} e^{h\sigma} \alpha(\mathrm{d}\sigma)$$

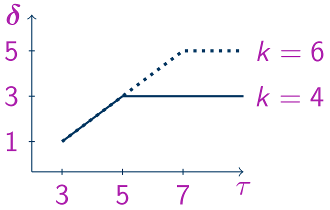
For $h > 0$ still one positive solution $m^+(\theta, h)$

Define the critical exponent $\boldsymbol{\delta}$ as

$$m^+(\theta_c, h) \asymp h^{1/\delta} \qquad \text{for } h \searrow 0$$

Theorem

$$\boldsymbol{\delta} = \begin{cases} k-1 & \text{for } \mathbb{E}[W^k] < \infty \\ \tau - 2 & \text{for } \tau \in (3, k+1) \end{cases}$$



Mean-field exponents

For $k = 4$ and $\mathbb{E}[W^4] < \infty$ we get the values

$$\beta = 1/2 \qquad \textit{and} \qquad \delta = 3$$

These are called *mean-field* values. They are the same for

- Curie-Weiss model
- Ising model on $\mathbb{Z}^d$, $d > 4$
  Aizenman, Fernández, '86
- Many other models

Note that these values do *not* hold for $\tau \leq 5$ or other values of k

Despite the fact that these are still a mean-field models!

Example models

Models with $k = 4$ for example include

- Ising model $\alpha = \frac{1}{2}(\delta_{-1} + \delta_1)$
  Giardinà, Giberti, van der Hofstad, Prioriello, '16;
  D.,Giardinà, Giberti, van der Hofstad, Prioriello, '16
- Beta distributions, $b > 0$

$$\alpha(\mathrm{d}\sigma) = \frac{1}{2B(b,b)} \left(\frac{1+\sigma}{2}\right)^{b-1} \left(\frac{1-\sigma}{2}\right)^{b-1} \mathrm{d}\sigma$$

- α uniform on $\mathbb{S}^q$ (similar to Beta model with $b = q/2$)

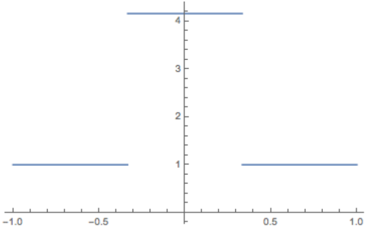
Example models

Models with $k = 4$ for example include

- Ising model $\alpha = \frac{1}{2}(\delta_{-1} + \delta_1)$
- Beta distributions, $\alpha(\mathrm{d}\sigma) = \frac{1}{2B(b,b)} \left(\frac{1+\sigma}{2}\right)^{b-1} \left(\frac{1-\sigma}{2}\right)^{b-1} \mathrm{d}\sigma, b > 0$
- α uniform on $\mathbb{S}^q$ (similar to Beta model with $b = q/2$)

Model with $k = 6$ where $\alpha_0(\mathrm{d}\sigma)$ equals

$$\frac{\mathrm{d}\sigma}{z} \begin{cases} 1 & \text{for } |\sigma| > \frac{1}{3} \\ 2(59 - 18\sqrt{10}) & \text{for } |\sigma| \leq \frac{1}{3} \end{cases}$$



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Related / future research

Can we find models with even higher values of k?

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Results on critical behavior for Ising model also hold in *quenched* case
D., Giardinà, van der Hofstad, '14
Can we prove similar resuls for general spin models in *quenched* case?

Related / future research

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Can we find models with even higher values of k?

Results on critical behavior for Ising model also hold in *quenched* case
D., Giardinà, van der Hofstad, '14
Can we prove similar resuls for general spin models in *quenched* case?

Can we prove CLTs for the total spin?
(As was done for Ising Giardinà, Giberti, van der Hofstad, Prioriello, '16)

Truncated moments

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For $a > \tau - 1$

$$\mathbb{E}[W^a \mathbb{1}_{\{W \le \ell\}}] \sim \int_1^\ell w^a w^{-\tau} \mathrm{d}w \sim \ell^{a-(\tau-1)}$$

Similarly, for $a < \tau - 1$

$$\mathbb{E}[W^a \mathbb{1}_{\{W > \ell\}}] \sim \int_\ell^\infty w^a w^{-\tau} \mathrm{d}w \sim \ell^{a-(\tau-1)}$$

Optimal choice is $\ell = 1/m^+$