

# Metastability of the Ising model on random regular graphs at zero temperature

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Construction of *random  $r$ -regular graph* with  $n$  vertices:

- ▶ *Assign*  $r$  half-edges to each vertex  $i \in \{1, \dots, n\}$
- ▶ *Attach* first half-edge to another half-edge *uniformly at random*
- ▶ Continue until all half-edges are connected

Denote resulting graph by  $G_n = (\{1, \dots, n\}, E_n)$

Ising measure on  $G_n$  for  $\sigma \in \{-1, +1\}^n$

$$\mu(\sigma) = \frac{1}{Z_n} e^{-\beta H(\sigma)}$$

with Hamiltonian

$$H(\sigma) = - \sum_{(i,j) \in E_n} \sigma_i \sigma_j - h \sum_{i=1}^n \sigma_i$$

where

- $\beta \geq 0$  inverse temperature
- $h > 0$  external magnetic field
- $Z_n$  normalization factor (partition function)

Expressions known for pressure, magnetization, etc.

Dembo, Montanari 2010; D., van der Hofstad, Giardinà 2010

Critical temperature  $\tanh \beta_c = \frac{1}{r-1}$

Critical exponents are mean field

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Proofs use

- ▶ locally tree-like structure of random graphs
- ▶ correlation inequalities to reduce graph problem to tree problem
- ▶ analysis of a tree-recursion

Discrete time *Markov chain* where at every time step

1) Pick vertex  $i$  uniformly from  $\{1, \dots, n\}$

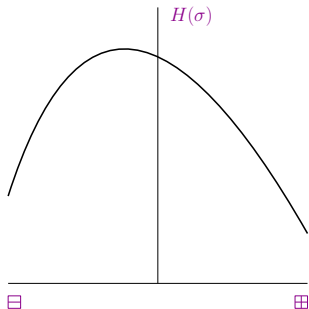
2) Flip spin  $\sigma_i$  with probability 
$$\begin{cases} 1 & \text{if } H(\sigma^i) \leq H(\sigma) \\ e^{-\beta[H(\sigma^i) - H(\sigma)]} & \text{if } H(\sigma^i) > H(\sigma) \end{cases}$$

Equilibrium distribution  $\mu(\sigma) = \frac{1}{Z_n} e^{-\beta H(\sigma)}$

Take zero temperature limit  $\beta \rightarrow \infty$

Then,  $\mu(\sigma) = \frac{1}{Z_n} e^{-\beta H(\sigma)}$  concentrates on minimizer of

$$H(\sigma) = - \sum_{(i,j) \in E_n} \sigma_i \sigma_j - h \sum_{i=1}^n \sigma_i$$



Minimizer is  $\blacksquare \rightarrow$  *stable* state

If  $h$  small, local minimum  $\square$   
(Usually) *metastable* state

Time  $\tau(\square, \blacksquare)$  to go from  $\square$  to  $\blacksquare$ ?

## Theorem

For random  $r$ -regular graphs with  $r \geq 3$ ,

$h \leq C_0 \sqrt{r}$  for small constant  $C_0$ ,

$\exists$  constants  $0 < C_1 < \infty, C_2 < \infty$  such that, whp,

$$\lim_{\beta \rightarrow \infty} \mathbb{P}[\exp\{\beta(r/2 - C_1 \sqrt{r})n\} < \tau(\boxminus, \boxplus) < \exp\{\beta(r/2 + C_2 \sqrt{r})n\}] = 1$$

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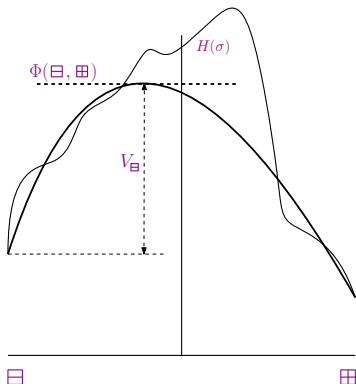
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Note that exponent is *linear* in  $n$

Big difference with e.g. lattices where  $\tau(\boxminus, \boxplus) \sim e^{\beta C}$

Cassandro, Galves, Olivieri, Vares 1984, Neves, Schonmann 1991  
Manzo, Nardi, Olivieri, Scoppola 2004, Cirillo, Nardi 2013



Communication height

$$\Phi(\sigma, \sigma') = \min_{\omega \text{ path from } \sigma \text{ to } \sigma'} \max_{\sigma'' \in \omega} H(\sigma'')$$

Stability level

$$V_{\sigma} = \min_{\sigma': H(\sigma') < H(\sigma)} \Phi(\sigma, \sigma') - H(\sigma)$$

## Proposition

If there exist  $0 < \Gamma_\ell \leq \Gamma_u < \infty$  such that

$$1) \Phi(\Xi, \Theta) - H(\Xi) \geq \Gamma_\ell$$

$$2) \Phi(\Xi, \Theta) - H(\Xi) \leq \Gamma_u$$

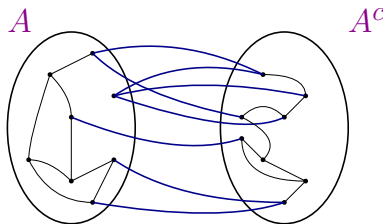
$$3) \text{ for all } \sigma \notin \{\Xi, \Theta\} \text{ it holds that } V_\sigma \leq \Gamma_u$$

then, for all  $\varepsilon > 0$ ,

$$\lim_{\beta \rightarrow \infty} \mathbb{P}[\exp\{\beta(\Gamma_\ell - \varepsilon)\} < \tau(\Xi, \Theta) < \exp\{\beta(\Gamma_u + \varepsilon)\}] = 1$$

The (edge) isoperimetric number of  $G_N$  is defined as

$$i_e(G_N) = \min_{\substack{A \subseteq \{1, \dots, n\} \\ |A| \leq n/2}} \frac{|\partial_e A|}{|A|}$$



Note that, for all  $A$  with  $|A| \leq n/2$ ,

$$|\partial_e A| \geq i_e(G_N)|A|$$

# Lower bound on $i_e$ for regular graphs

Lemma (Bollobás 1988)

For  $r$ -regular random graphs with  $r \geq 3$ , whp,

$$i_e(G_n) \geq \frac{r}{2} - \sqrt{\log 2} \sqrt{r}$$

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Note that  $i_e$  is bounded away from 0 for  $n \rightarrow \infty$

Such graphs are called *expander graphs*

This in contrast to lattices for which  $i_e \rightarrow 0$  as  $n \rightarrow \infty$

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Proof by bounding probability that given set has a small boundary and then using union bound

# Upper bound on $i_e$ for regular graphs

Lemma (Alon 1997)

$\exists C > 0$  such that, for all  $r$ -regular graphs with  $n \geq 40r^9$ ,

$\exists A \subset [n]$  with  $|A| = \lfloor n/2 \rfloor$  such that

$$\frac{|\partial_e A|}{|A|} \leq \frac{r}{2} - C\sqrt{r}$$

and hence also

$$i_e(G_n) \leq \frac{r}{2} - C\sqrt{r}$$

Condition 1)  $\Phi(\boxminus, \boxplus) - H(\boxminus) \geq \Gamma_\ell$

### Lemma

If  $h \leq C_0 \sqrt{r}$  for small constant  $C_0$ ,

then  $\exists C_1 > 0$  such that  $\forall \sigma$  with  $n/2 + \text{spins}$

$$H(\sigma) - H(\boxminus) \geq (i_e(G_n) - h)n \geq \left(\frac{r}{2} - C_1 \sqrt{r}\right) n$$

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Proof.



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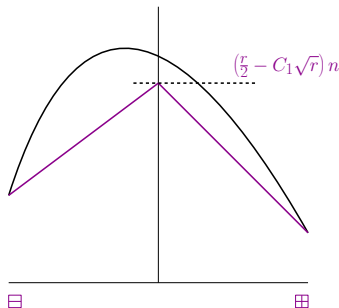
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Similarly one can show that any path must lie above this purple line:

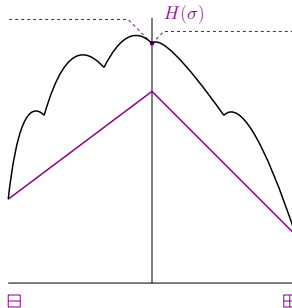


## Condition 2) $\Phi(\boxminus, \boxplus) - H(\boxminus) \leq \Gamma_u$

Use that there exists a configuration with  $n/2 +$  spins, call this the set  $A$ , with

$$|\partial_e A| \leq \left(\frac{r}{2} - C\sqrt{r}\right) \frac{n}{2}$$

and that there always exists a path to  $\boxminus$  and  $\boxplus$  that doesn't increase energy too much



# Going to lower energy configurations

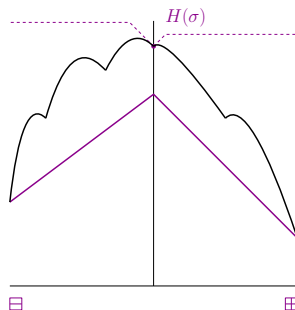
Start with set of plus spins  $A$  with  $|A| \leq n/2$

Use the following (implicit) construction:

1. Select a  $+$  spin with at least 1  
– neighbor and flip this spin  
Energy goes up at most  $2(r - 2 + h)$
2. Repeat this  $s$  times (choose  $s$  later)
3. Select a  $+$  spin with at least  $(r + h)/2$   
– neighbors and flip this spin  
Energy doesn't go up
4. Repeat this until no such spins are left

Choose  $s$  big enough so that energy at end  
is guaranteed to be smaller than at start

Energy barrier between start and end is at most  $2(r - 2 + h)s$



# Going to lower energy configurations

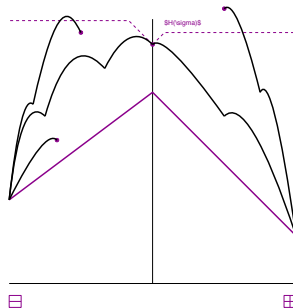
One can choose  $s = \mathcal{O}(n/\sqrt{r})$ , so that energy barrier is at most

$$2(r - 2 + h)s = \mathcal{O}(\sqrt{rn}).$$

This proves Condition 2) because indeed

$$\Phi(\boxplus, \boxplus) - H(\boxplus) \leq \left(\frac{r}{2} + C_2\sqrt{r}\right) \frac{n}{2}$$

Same argument can be used for Condition 3)



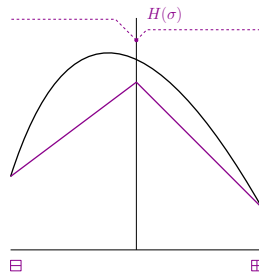
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# $\boxplus$ is the metastable state

## Definition

$\eta$  is a metastable state if

$$V_\eta = \max_{\sigma \neq \boxplus} V_\sigma$$

## Theorem

For random  $r$ -regular graphs with  $r \geq 6$ ,

$h \leq C_0 \sqrt{r}$  for small constant  $C_0$ ,

$\boxplus$  is the unique metastable state whp

# Proof that $\boxminus$ is metastable

## Proposition

*If there exist  $0 < \Gamma_\ell < \infty$  such that*

$$1) \Phi(\boxminus, \boxplus) - H(\boxminus) \geq \Gamma_\ell$$

*3') for all  $\sigma \notin \{\boxminus, \boxplus\}$  it holds that  $V_\sigma < \Gamma_\ell$*

*then  $\boxminus$  is the unique metastable state*

# Proof that $\square$ is metastable

One has to prove that

$$2(r - 2 + h) \frac{C'}{\sqrt{r}} n < (r/2 - C_1 \sqrt{r}) n$$

We need to compare constants (and need to improve them for  $r \leq 10$ )

# Proof that $\square$ is metastable

One has to prove that

$$2(r - 2 + h) \frac{C'}{\sqrt{r}} n < (r/2 - C_1 \sqrt{r}) n$$

We need to compare constants (and need to improve them for  $r \leq 10$ )

For this note that for  $r \geq 6$  we have that  $i_e > 1$

Hence in any set  $A$  with  $|A| \leq n/2$  there exists a vertex with at least 2 neighbors in  $A^c$

This improves the bounds in the proof of Condition 3)  
which are sufficient to prove Condition 3')

Can we prove that  $\tau(\boxminus, \boxplus) \sim \exp\{\beta \Gamma_n\}$  with

$$\lim_{n \rightarrow \infty} \frac{\Gamma_n}{n} = \gamma$$

and determine  $\gamma$ ?

What do the critical droplet and tube of typical trajectories look like?

What about positive temperature? (In the limit  $h \searrow 0$ , or  $n \rightarrow \infty$ )?

What about more general degrees? (next talk)