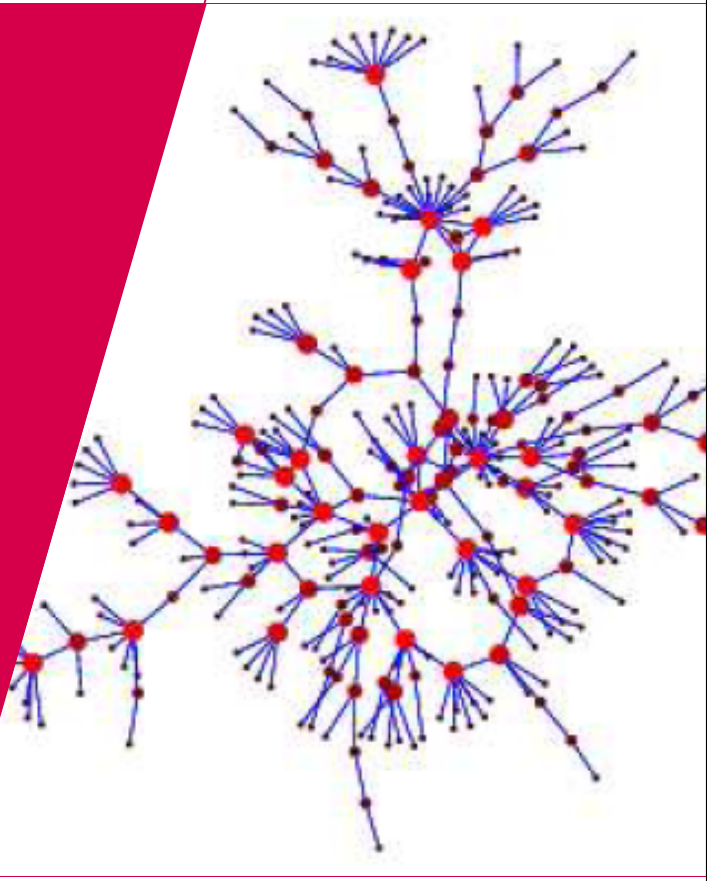
Ising models on power-law random graphs

Sander Dommers

Joint work with:
Cristian Giardinà
Remco van der Hofstad



Technische Universiteit
Eindhoven
University of Technology

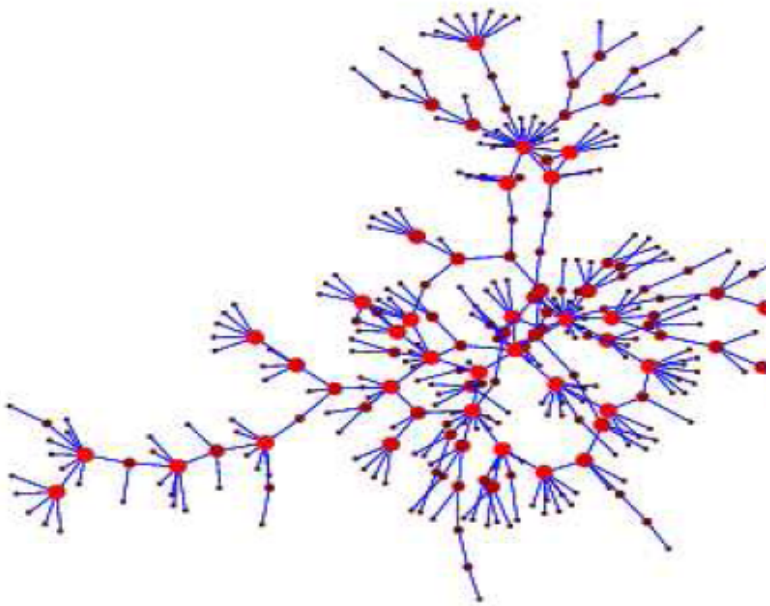
Where innovation starts

Statistical Mechanics on Random Structures
November 17, 2009

Introduction

There are many complex real-world networks, e.g.,

- Social networks (friendships, business relationships, *sexual contacts,* ...);
- Information networks (World Wide Web, citations, ...);
- Technological networks (Internet, airline routes, ...);
- Biological networks (protein interactions, neural networks,...).

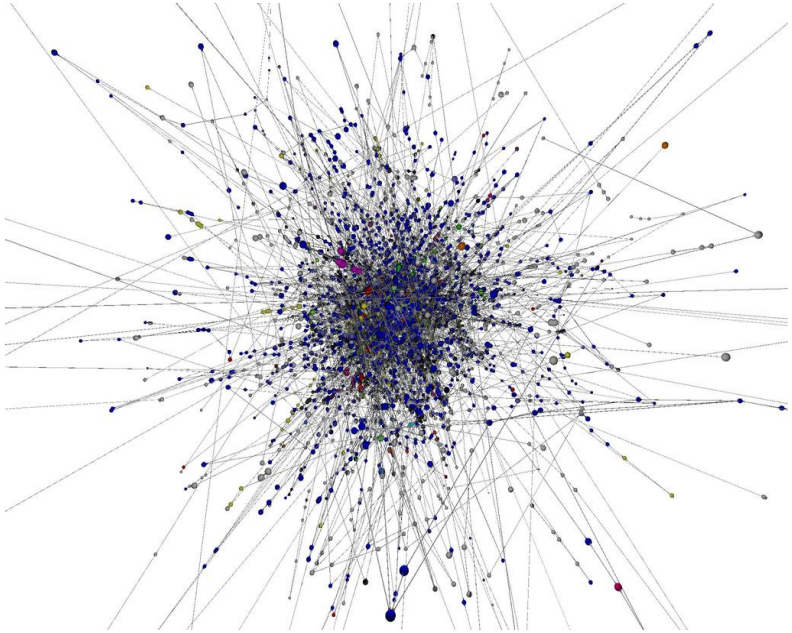


Sexual network Colorado Springs, USA
(Potterat, et al., '02)

Introduction

There are many complex real-world networks, e.g.,

- Social networks (friendships, business relationships, sexual contacts, ...);
- Information networks (World Wide Web, citations, ...);
- Technological networks (*Internet,* airline routes, ...);
- Biological networks (protein interactions, neural networks,...).

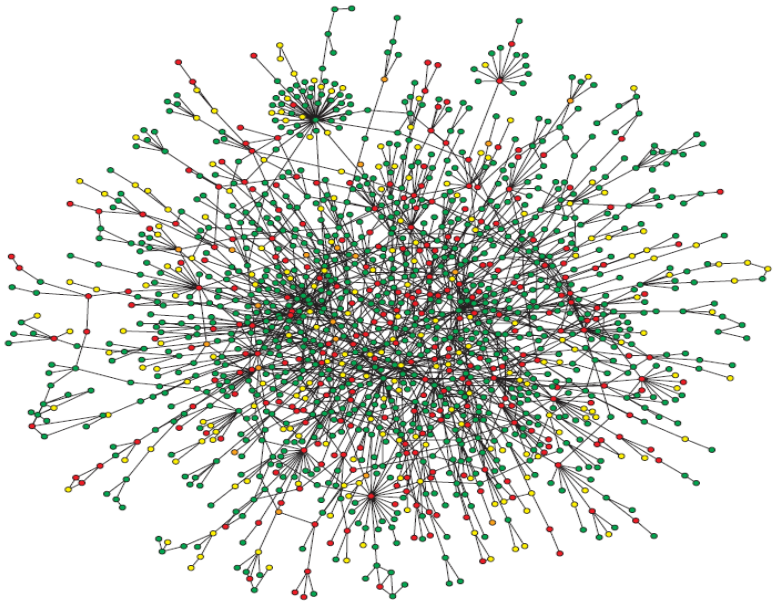


Small part of the Internet (http://www.fractalus.com/steve/stuff/ipmap/)

Introduction

There are many complex real-world networks, e.g.,

- Social networks (friendships, business relationships, sexual contacts, ...);
- Information networks (World Wide Web, citations, ...);
- Technological networks (Internet, airline routes, ...);
- Biological networks (*protein interactions,* neural networks,...).

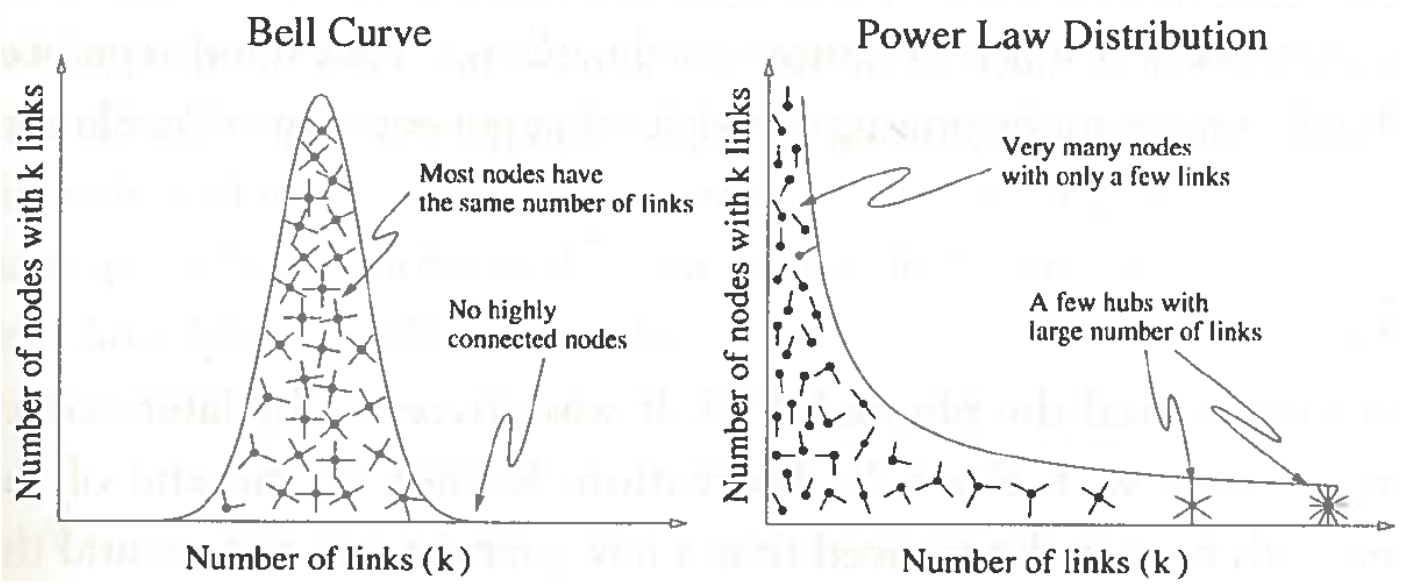


Yeast protein interaction network
(Jeong, et al., '01)

Properties of complex networks

Power-law behavior

Number of vertices with degree k is proportional to $k^{-\tau}$.
Often, $2 < \tau < 3$ (finite mean, infinite variance).



Barabási, Linked, '02

Small worlds

Distances in the network are small

Ising model

Ising model: paradigm model in statistical physics for *cooperative behavior.*

When studied on complex networks it can model for example *opinion spreading* in society.

What are effects of *structure* of complex networks on *behavior* of Ising model?

Power-law random graphs

In the *configuration model* a graph $G_n = (V_n = [n], E_n)$ is constructed as follows.

- Let D have a certain distribution (the *degree distribution*);
- *Assign* D_i half-edges to each vertex $i \in [n]$, where D_i are i.i.d. like D (Add one half-edge to last vertex when the total number of half-edges is odd);
- *Attach* first half-edge to another half-edge *uniformly at random;*
- Continue until all half-edges are connected.

Special attention to power-law degree sequences, i.e.,

$$\mathbb{P}[D \geq k] \leq ck^{-(\tau-1)}, \qquad \tau > 1.$$

Local structure configuration model for $\tau > 2$

Start from random vertex i which has degree D_i.

Look at neighbors of vertex i, probability such a neighbor has degree $k+1$ is approximately,

$$\frac{(k+1)\sum_{j\in[n]} \mathbb{1}_{\{D_j=k+1\}}}{\sum_{j\in[n]} D_j}$$

Local structure configuration model for $\tau > 2$

Start from random vertex i which has degree D_i.

Look at neighbors of vertex i, probability such a neighbor has degree $k+1$ is approximately,

$$\frac{(k+1)\sum_{j\in[n]} \mathbb{1}_{\{D_j=k+1\}}/n}{\sum_{j\in[n]} D_j/n} \longrightarrow \frac{(k+1)\mathbb{P}[D=k+1]}{\mathbb{E}[D]}, \qquad \text{for } \tau > 2.$$

Local structure configuration model for $\tau > 2$

Start from random vertex i which has degree D_i.

Look at neighbors of vertex i, probability such a neighbor has degree $k+1$ is approximately,

$$\frac{(k+1)\sum_{j\in[n]} \mathbb{1}_{\{D_j=k+1\}}/n}{\sum_{j\in[n]} D_j/n} \longrightarrow \frac{(k+1)\mathbb{P}[D=k+1]}{\mathbb{E}[D]}, \qquad \text{for } \tau > 2.$$

Let K have distribution (the *forward degree* distribution),

$$\mathbb{P}[K=k] = \frac{(k+1)\mathbb{P}[D=k+1]}{\mathbb{E}[D]}.$$

Locally tree-like structure: a branching process with offspring D in first generation and K in further generations. Also, *uniformly sparse.*

Definition of the Ising model

On a graph G_n, the *ferromagnetic Ising model* is given by the following Boltzmann distributions over $\sigma \in \{-1, +1\}^n$,

$$\mu(\sigma) = \frac{1}{Z_n(\beta, B)} \exp\left\{\beta \sum_{(i,j)\in E_n} \sigma_i \sigma_j + \sum_{i\in[n]} B\sigma_i\right\},$$

where

- $\beta \geq 0$ is the inverse temperature;
- B is the external magnetic field;
- $Z_n(\beta, B)$ is a normalization factor (the *partition function*).

Critical temperature

Define the *magnetization* on G_n as

$$m_n(\beta, B) = \Big\langle \frac{1}{n} \sum_{i=1}^{n} \sigma_i \Big\rangle_\mu .$$

Then, the *spontaneous magnetization,*

$$M = \lim_{B \downarrow 0} \lim_{n \to \infty} m_n \begin{cases} = 0, & \beta < \beta_c; \\ > 0, & \beta > \beta_c. \end{cases}$$

The *critical inverse temperature* β_c is given by

$$\mathbb{E}[K](\tanh \beta_c) = 1.$$

Note that, for $\tau \in (2, 3)$, we have $\mathbb{E}[K] = \infty$, so that $\beta_c = 0$.

Pressure in thermodynamic limit ($\mathbb{E}[K] < \infty$)

Theorem (Dembo, Montanari, '08)

For a locally tree-like and uniformly sparse graph sequence $\{G_n\}_{n\geq 1}$ with $\mathbb{E}[K] < \infty$, the pressure per particle,

$$\psi_n(\beta, B) = \frac{1}{n} \log Z_n(\beta, B),$$

converges, for $n \to \infty$, to

$$\varphi_h(\beta, B) \equiv \frac{\mathbb{E}[D]}{2} \log\cosh(\beta) - \frac{\mathbb{E}[D]}{2}\mathbb{E}[\log(1 + \tanh(\beta)\tanh(h_1)\tanh(h_2))]$$
$$+ \mathbb{E}\left[\log\left(e^{B}\prod_{i=1}^{D}\{1 + \tanh(\beta)\tanh(h_i)\}\right.\right.$$
$$\left.\left. + e^{-B}\prod_{i=1}^{D}\{1 - \tanh(\beta)\tanh(h_i)\}\right)\right].$$

Pressure in thermodynamic limit ($\mathbb{E}[D] < \infty$)

Theorem (DGvdH, '09)

Let $\tau > 2$. Then, in the configuration model, the pressure per particle,

$$\psi_n(\beta, B) = \frac{1}{n} \log Z_n(\beta, B),$$

converges almost surely, for $n \to \infty$, to

$$\varphi_h(\beta, B) \equiv \frac{\mathbb{E}[D]}{2} \log\cosh(\beta) - \frac{\mathbb{E}[D]}{2} \mathbb{E}[\log(1 + \tanh(\beta)\tanh(h_1)\tanh(h_2))]$$
$$+ \mathbb{E}\left[\log\left(e^B \prod_{i=1}^{D} \{1 + \tanh(\beta)\tanh(h_i)\}\right.\right.$$
$$\left.\left. + e^{-B} \prod_{i=1}^{D} \{1 - \tanh(\beta)\tanh(h_i)\}\right)\right].$$

Tree recursion

Proposition

Let K_t be i.i.d. like K and $B > 0$. Then, the recursion

$$h^{(t+1)} \stackrel{d}{=} B + \sum_{i=1}^{K_t} \operatorname{atanh}(\tanh(\beta)\tanh(h_i^{(t)})),$$

has a unique fixed point h_β^*.

Interpretation: the *effective field* of a vertex in a *tree* expressed in that of its neighbors.

Uniqueness shown by showing that effect of *boundary conditions* on generation t vanishes for $t \to \infty$.

This is done using *monotonicity* in β and B and *concavity* in B of the magnetization in the ferromagnetic Ising model.

Outline of the proof

$$\lim_{n\to\infty} \psi_n(\beta, B)$$

$$= \lim_{\varepsilon\downarrow 0} \lim_{n\to\infty} \left[\psi_n(0, B) + \int_0^{\varepsilon} \frac{\partial}{\partial \beta'} \psi_n(\beta', B) \mathrm{d}\beta' + \int_{\varepsilon}^{\beta} \frac{\partial}{\partial \beta'} \psi_n(\beta', B) \mathrm{d}\beta' \right]$$

$$= \varphi_h(0, B) + 0 + \lim_{\varepsilon\downarrow 0} \int_{\varepsilon}^{\beta} \frac{\partial}{\partial \beta'} \varphi(\beta', B) \mathrm{d}\beta'$$

$$= \varphi_h(\beta, B).$$

Outline of the proof

$$\lim_{n\to\infty} \psi_n(\beta, B)$$

$$= \lim_{\varepsilon\downarrow 0} \lim_{n\to\infty} \left[\psi_n(0, B) + \int_0^{\varepsilon} \frac{\partial}{\partial \beta'} \psi_n(\beta', B) \mathrm{d}\beta' + \int_{\varepsilon}^{\beta} \frac{\partial}{\partial \beta'} \psi_n(\beta', B) \mathrm{d}\beta' \right]$$

$$= \varphi_h(0, B) + 0 + \lim_{\varepsilon\downarrow 0} \int_{\varepsilon}^{\beta} \frac{\partial}{\partial \beta'} \varphi(\beta', B) \mathrm{d}\beta'$$

$$= \varphi_h(\beta, B).$$

Outline of the proof

$$\lim_{n\to\infty} \psi_n(\beta, B)$$

$$= \lim_{\varepsilon\downarrow 0} \lim_{n\to\infty} \left[\psi_n(0, B) + \int_0^{\varepsilon} \frac{\partial}{\partial\beta'}\psi_n(\beta', B)\mathrm{d}\beta' + \int_{\varepsilon}^{\beta} \frac{\partial}{\partial\beta'}\psi_n(\beta', B)\mathrm{d}\beta'\right]$$

$$= \varphi_h(0, B) + 0 + \lim_{\varepsilon\downarrow 0} \int_{\varepsilon}^{\beta} \frac{\partial}{\partial\beta'}\varphi(\beta', B)\mathrm{d}\beta'$$

$$= \varphi_h(\beta, B).$$

Outline of the proof

$$\lim_{n\to\infty} \psi_n(\beta, B)$$

$$= \lim_{\varepsilon\downarrow 0} \lim_{n\to\infty} \left[\psi_n(0, B) + \int_0^{\varepsilon} \frac{\partial}{\partial \beta'} \psi_n(\beta', B) \mathrm{d}\beta' + \int_{\varepsilon}^{\beta} \frac{\partial}{\partial \beta'} \psi_n(\beta', B) \mathrm{d}\beta' \right]$$

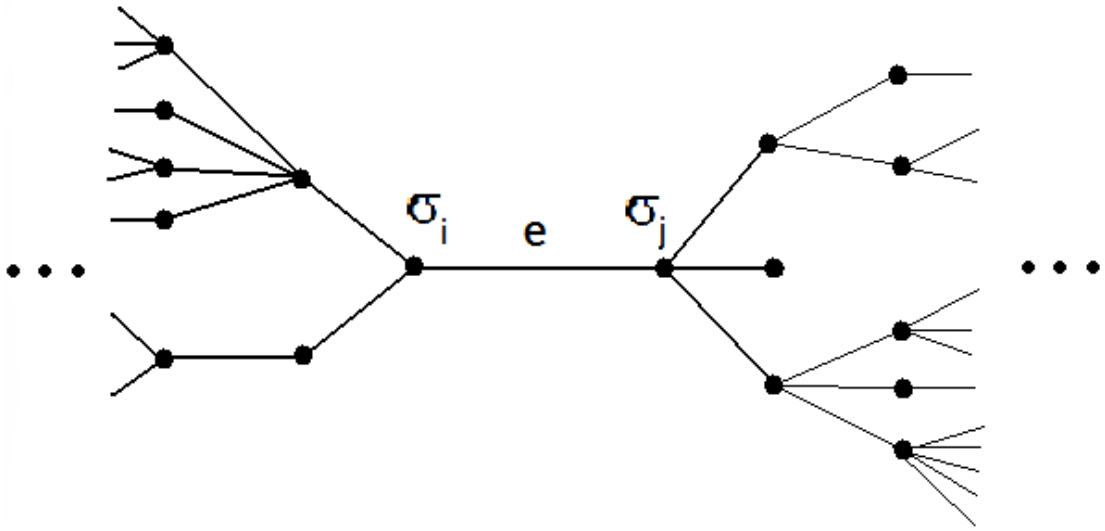
$$= \varphi_h(0, B) + 0 + \lim_{\varepsilon\downarrow 0} \int_{\varepsilon}^{\beta} \frac{\partial}{\partial \beta'} \varphi(\beta', B) \mathrm{d}\beta'$$

$$= \varphi_h(\beta, B).$$

Derivative of pressure

$$\frac{\partial}{\partial\beta}\psi_n(\beta, B) = \frac{1}{n}\sum_{(i,j)\in E_n}\langle\sigma_i\sigma_j\rangle_\mu = \frac{|E_n|}{n}\frac{\sum_{(i,j)\in E_n}\langle\sigma_i\sigma_j\rangle_\mu}{|E_n|}$$

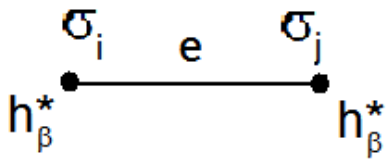
$$\longrightarrow \frac{\mathbb{E}[D]}{2}\mathbb{E}\left[\langle\sigma_i\sigma_j\rangle_\mu\right]$$



Derivative of pressure

$$\frac{\partial}{\partial\beta}\psi_n(\beta, B) = \frac{1}{n}\sum_{(i,j)\in E_n}\langle\sigma_i\sigma_j\rangle_\mu = \frac{|E_n|}{n}\frac{\sum_{(i,j)\in E_n}\langle\sigma_i\sigma_j\rangle_\mu}{|E_n|}$$

$$\longrightarrow \frac{\mathbb{E}[D]}{2}\mathbb{E}\left[\langle\sigma_i\sigma_j\rangle_\mu\right]$$



$$\frac{\mathbb{E}[D]}{2}\mathbb{E}\left[\langle\sigma_i\sigma_j\rangle_\mu\right] \longrightarrow \frac{\mathbb{E}[D]}{2}\mathbb{E}\left[\langle\sigma_i\sigma_j\rangle_e\right]$$

Derivative of φ

$$\frac{\partial}{\partial\beta}\varphi_{h^*_\beta}(\beta, B) = \frac{\mathbb{E}[D]}{2}\mathbb{E}\left[\left\langle\sigma_i\sigma_j\right\rangle_e\right].$$

$$\varphi_h(\beta, B) = \frac{\mathbb{E}[D]}{2}\log\cosh(\beta) - \frac{\mathbb{E}[D]}{2}\mathbb{E}[\log(1+\tanh(\beta)\tanh(h_1)\tanh(h_2))]$$
$$+\mathbb{E}\left[\log\left(e^B\prod_{i=1}^{D}\{1+\tanh(\beta)\tanh(h_i)\} + e^{-B}\prod_{i=1}^{D}\{1-\tanh(\beta)\tanh(h_i)\}\right)\right]$$

- Show that we can ignore dependence of h^*_β on β; (*Interpolation* techniques. Split analysis into two parts, one for *small degrees* and one for *large degrees*)
- Compute the derivative with assuming β fixed in h^*_β.

Distances in power-law random graphs

Let H_n be the graph distance between two *uniformly chosen connected* vertices in the configuration model. Then:

- For $\tau > 3$ and $\mathbb{E}[K] > 1$ (vdH, Hooghiemstra, Van Mieghem, '05),

$$H_n \sim \log n,$$

- For $\tau \in (2, 3)$ (vdH, Hooghiemstra, Znamenski, '07),

$$H_n \sim \log \log n;$$

Distances in power-law random graphs

Let H_n be the graph distance between two *uniformly chosen connected* vertices in the configuration model. Then:

- For $\tau > 3$ and $\mathbb{E}[K] > 1$ (vdH, Hooghiemstra, Van Mieghem, '05),

$$H_n \sim \log n,$$

- For $\tau \in (2, 3)$ (vdH, Hooghiemstra, Znamenski, '07),

$$H_n \sim \log \log n;$$

For $\tau > 3$ and $\tau \in (2, 3)$ similar results hold for the diameter of linear preferential attachment models (D, vdH, Hooghiemstra, '09).

Critical exponents

Predictions by physicists (Dorogovtsev, Goltsev, Mendes, '02 and Leone, Vázquez, Vespignani, Zecchina, '02).

Critical behavior for $\beta \downarrow \beta_c$ of *magnetization* m, *specific heat* δC and *susceptibility* χ.

	m	δC	χ
$\tau > 5$	$\sim (\beta - \beta_c)^{1/2}$	jump at β_c	$\sim (\beta - \beta_c)^{-1}$
$\tau \in (3, 5)$	$\sim (\beta - \beta_c)^{1/(\tau-3)}$	$\sim (\beta - \beta_c)^{(5-\tau)/(\tau-3)}$	
$\tau \in (2, 3)$	$\sim (\beta - \beta_c)^{1/(3-\tau)}$	$\sim (\beta - \beta_c)^{(\tau-1)/(3-\tau)}$	$\sim (\beta - \beta_c)^{1}$