Wiederholung

Wellengleichung: $\vec{n}\times(\vec{n}\times\vec{E})+K\cdot\vec{E}=0$ mit $\vec{n}=\frac{\vec{k}\,c}{\omega}$

Wellenvektor in x-z-Ebene:

$$\begin{pmatrix} S-n^2\cos^2\theta & -iD & n^2\cos\theta\sin\theta \\ iD & S-n^2 & 0 \\ n^2\cos\theta\sin\theta & 0 & P-n^2\sin^2\theta \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}=0$$

Existenz einer Lösung: $A\,n^4-B\,n^2+C=0$

Mit

$$A=S\sin^2\theta+P\cos^2\theta$$

$$B=(-S^2+D^2)\sin^2\theta+PS(1+\cos^2\theta)=RL\sin^2\theta+PS(1+\cos^2\theta)$$

$$C=(S^2-D^2)P=\left(\frac{1}{4}(R+L)^2+\frac{1}{4}(R-L)^2\right)P=4\,RL\frac{P}{4}=PRL$$

Wiederholung

$$R = K_1 + iK_2 = 1 - \sum_j \frac{\omega_{pj}^2}{\omega\left(\omega + \epsilon_j \omega_{cj}\right)}$$

$$L = K_1 - iK_2 = 1 - \sum_j \frac{\omega_{pj}^2}{\omega\left(\omega - \epsilon_j \omega_{cj}\right)}$$

$$S = \frac{1}{2}\left(R + L\right) = K_1 = 1 - \sum_j \frac{\omega_{pj}^2}{\omega^2 - \omega_{cj}^2}$$

$$D = \frac{1}{2}\left(R - L\right) = iK_2 = 1 - \sum_j \frac{\epsilon_j \omega_{cj} \omega_{pj}^2}{\omega\left(\omega^2 - \omega_{cj}^2\right)}$$

$$P = K_3 = 1 - \sum_j \frac{\omega_{pj}^2}{\omega^2}$$

Dispersionsrelation

$$n^4 - \frac{B}{A} n^2 + \frac{C}{A} = 0$$

Quadratische Gleichung in n²

→ Lösung:

$$n^2 = \frac{B}{2A} \pm \sqrt{\frac{B^2}{4A^2} - \frac{C}{A}} = \frac{B \pm F}{2A} \rightarrow F^2 = B^2 - 4AC$$

$$B^2 - 4AC = \left(RLsin^2\theta + PS\left(1+\cos^2\theta\right)\right)^2 - \left(4S\sin^2\theta + 4P\cos^2\theta\right) PRL$$

$$= R^2 L^2 \sin^4\theta + 2PSRL\sin^2\theta\left(2-\sin^2\theta\right) + P^2 S^2 \underbrace{\left(2-\sin^2\theta\right)^2}_{4-4\sin^2\theta+\sin^4\theta = 4\cos^2\theta+\sin^4\theta}$$

$$-4SPRL\sin^2\theta - 4P^2 RL\cos^2\theta$$

$$= \left(RL - PS\right)^2 \sin^4\theta + 4PSRL\sin^2\theta - 4PSRL\sin^2\theta$$

$$+ \underbrace{\left(4P^2 S^2 - 4P^2 RL\right)}_{4P^2\left(S^2-RL\right) = 4P^2\left(S^2-S^2+D^2\right)} \cos^2\theta$$

$$\rightarrow F^2 = \left(RL - PS\right)^2 \sin^4\theta + 4P^2 D^2 \cos^2\theta$$

Dispersionsrelation

Alternativ: Darstellung über Winkel:

$$A\,n^4 - B\,n^2 + C = 0$$

$$\left(S\sin^2\theta + P\cos^2\theta\right)n^4 - \left(RL\sin^2\theta + PS\underbrace{\left(1+\cos^2\theta\right)}_{\sin^2\theta + 2\cos^2\theta}\right)n^2 + PRL = 0$$

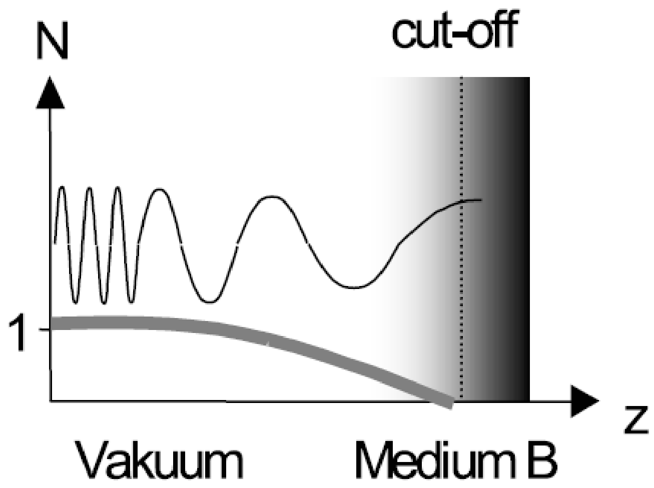
$$\sin^2\theta\left(Sn^4 + (RL - PS)\,n^2\right) + \cos^2\theta\left(Pn^4 - 2\mathrm{PS}\,n^2\right) + PRL\left(\sin^2\theta + \cos^2\theta\right) = 0$$

$$\sin^2\theta\left(Sn^4 + (RL - PS)\,n^2 + PRL\right) + \cos^2\theta\left(Pn^4 - 2\mathrm{PS}\,n^2 + PRL\right) = 0$$

$$\rightarrow \tan^2\theta = -\frac{P\left(n^4 - \overbrace{2\mathrm{S}}^{R+L}\,n^2 + RL\right)}{Sn^4 - (RL + PS)\,n^2 + PRL} = -\frac{P\left(n^2 - R\right)\left(n^2 - L\right)}{\left(Sn^2 - RL\right)\left(n^2 - P\right)}$$

Cutoff / Resonanz

$$N = \left|\frac{kc}{\omega}\right| \rightarrow 0, \lambda \rightarrow \infty, v_{ph} \rightarrow \infty$$

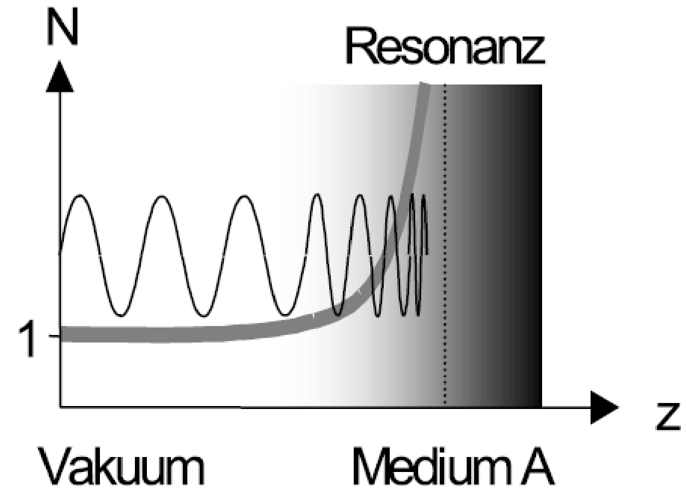


$$\rightarrow C = 0$$

$$\rightarrow PRL = 0$$

Allgemeine Cutoff-Bedingung

$$N = \left|\frac{kc}{\omega}\right| \rightarrow \infty, \lambda \rightarrow 0, v_{ph} \rightarrow 0$$



$$\rightarrow A = 0$$

$$\rightarrow \tan^2\theta = -\frac{P}{S}$$

Allgemeine Resonanz-Bedingung

Prinzipallösungen

Test: $B_0=0$, ODBA: $\theta=0$ $\Rightarrow D=0, R=L=S$

$$Pn^4-PSn^2+PS^2=0$$

1.) $P=0$

$$0=1-\sum_j \frac{\omega_{pj}^2}{\omega^2} \Leftrightarrow \omega^2=\sum_j \omega_{pj}^2$$

Plasma-Oszillation

2.) $P\neq 0$

$$n^4-2\mathrm{Sn}^2+S^2=0 \Rightarrow (n^2-S)^2=0$$

$$n^2=S \Rightarrow \frac{k^2c^2}{\omega^2}=1-\sum_j \frac{\omega_{pj}^2}{\omega^2}$$

Bzw. $\omega^2=k^2c^2+\sum_j \omega_{pj}^2$ Wie bereits bekannt.

Prinzipallösungen R/L-Welle

Ausbreitung parallel zu $B_0 \Leftrightarrow \theta = 0$

$$Pn^4 - PSn^2 + PRL = 0$$

1.) $P = 0$

$$0 = 1 - \sum_j \frac{\omega_{pj}^2}{\omega^2} \Leftrightarrow \omega^2 = \sum_j \omega_{pj}^2$$

Plasma-Oszillation

2.) $P \neq 0$

$$n^4 - 2\mathrm{S}n^2 + RL = 0$$

$$\Rightarrow n^2 = S \pm \sqrt{S^2 - RL}$$

$$RL = (S + D)(S - D) = S^2 - D^2$$

$$\Rightarrow n^2 = S \pm D = \begin{cases} R = 1 - \sum_j \dfrac{\omega_{pj}^2}{\omega(\omega + \epsilon_j \omega_{cj})} \\ L = 1 - \sum_j \dfrac{\omega_{pj}^2}{\omega(\omega - \epsilon_j \omega_{cj})} \end{cases}$$

R-Welle

L-Welle

Prinzipallösungen, R/L-Welle

Wellengleichung

$$\begin{pmatrix} S-n^2 & -iD & 0 \\ iD & S-n^2 & 0 \\ 0 & 0 & p-n^2 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

$$\begin{aligned} \left(S-n^2\right)E_x \quad & -iDE_y & =0 \\ iDE_x \quad & +\left(S-n^2\right)E_y & =0 \\ & \left(p-n^2\right)E_z & =0 \end{aligned}$$

$$\Rightarrow E_y = \frac{1}{2}\left(\frac{\left(S-n^2\right)^2 + D^2}{iD\left(S-n^2\right)}\right)E_x$$

Proportional, mit Phasenverschiebung → elliptisch polarisiert.

Prinzipallösungen R-Welle

Nur Elektronen und einfach geladene Ionen:

$$n_R^2 = R = 1 - \frac{\omega_{pi}^2}{\omega(\omega + \omega_{ci})} - \frac{\omega_{pe}^2}{\omega(\omega - \omega_{ce})}$$

Resonanz: $n^2 = R = \infty \Rightarrow \omega = \omega_{ce}$

Cutoff: $n^2 = R = 0$:

$$\omega_R = \frac{\omega_{ce} - \omega_{ci}}{2} + \left[\left(\frac{\omega_{ce} + \omega_{ci}}{2} \right)^2 + \omega_p^2 \right]^{\frac{1}{2}}$$

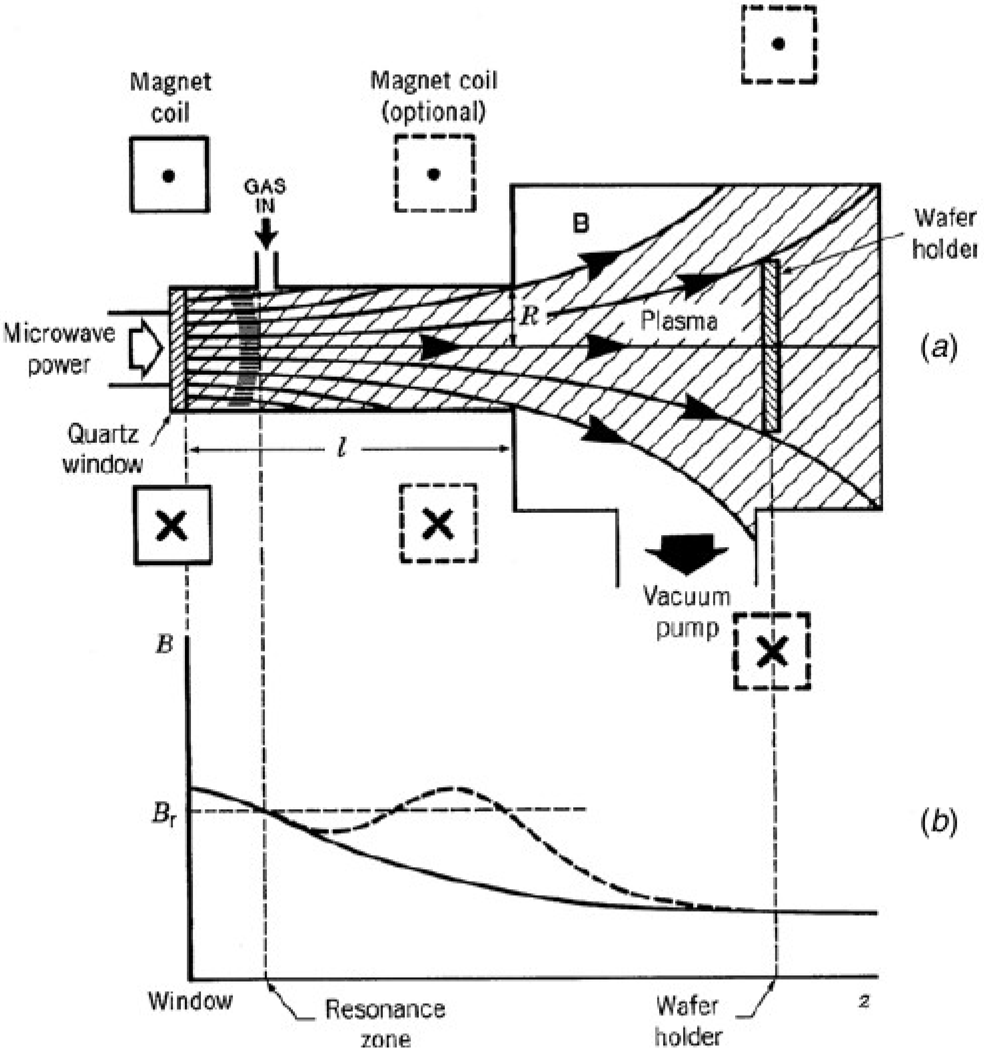
Mit: $\omega_p^2 = \omega_{pe}^2 + \omega_{pi}^2$

Näherungsweise, mit $m_e \ll m_i$: $\omega_R = \begin{cases} \omega_{ce}\left(1 + \dfrac{\omega_{pe}^2}{\omega_{ce}^2}\right) & \text{Cutoff, niedrige Dichte} \\ \omega_{pe} + \dfrac{1}{2}\omega_{ce} & \text{Cutoff, hohe Dichte} \end{cases}$

Prinzipallösungen R-Welle

Beispiel: ECRH-Entladung

$$n_R^2 = R = 1 - \frac{\omega_{pi}^2}{\omega(\omega + \omega_{ci})} - \frac{\omega_{pe}^2}{\omega(\omega - \omega_{ce})}$$



Prinzipallösungen R-Welle

Nur Elektronen und einfach geladene Ionen:

$$n_R^2 = R = 1 - \frac{\omega_{pi}^2}{\omega(\omega+\omega_{ci})} - \frac{\omega_{pe}^2}{\omega(\omega-\omega_{ce})}$$

Resonanz: $n^2 = R = \infty \Rightarrow \omega = \omega_{ce}$

Cutoff: $n^2 = R = 0:$

$$\omega_R = \frac{\omega_{ce}-\omega_{ci}}{2} + \left[\left(\frac{\omega_{ce}+\omega_{ci}}{2}\right)^2 + \omega_p^2\right]^{\frac{1}{2}}$$

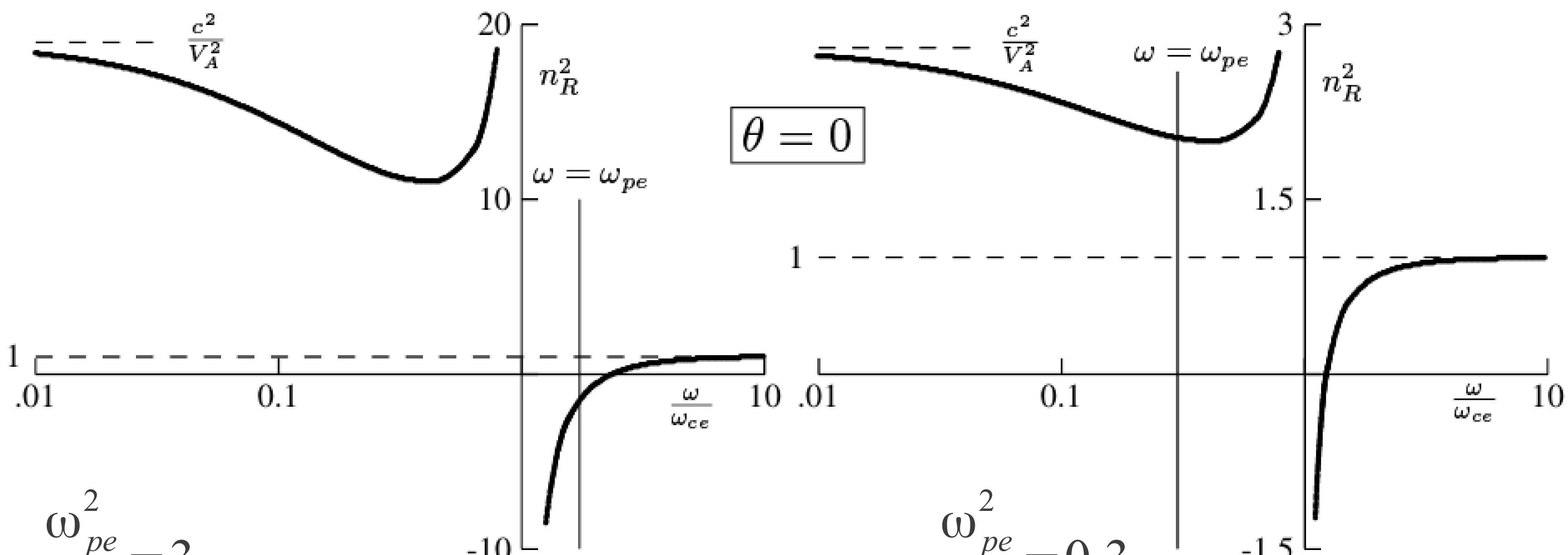
Mit: $\omega_p^2 = \omega_{pe}^2 + \omega_{pi}^2$

Näherungsweise, mit $m_e \ll m_i$: $\omega_R = \begin{cases} \omega_{ce}\left(1+\frac{\omega_{pe}^2}{\omega_{ce}^2}\right) & \text{Cutoff, niedrige Dichte} \\ \omega_{pe} + \frac{1}{2}\omega_{ce} & \text{Cutoff, hohe Dichte} \end{cases}$

Prinzipallösungen R-Welle

Für hohe / niedrige Frequenzen

$$n_R^2 = \begin{cases} 1 + \dfrac{\omega_{pi}^2}{\omega_{ci}^2} := \dfrac{c^2}{v_A^2} & \omega \to 0 \\[2ex] 1 - \dfrac{\omega_{pe}^2}{\omega^2} & \omega \to \infty \end{cases}$$



$$\frac{\omega_{pe}^2}{\omega_{ce}^2} = 3 \qquad \frac{\omega_{pe}^2}{\omega_{ce}^2} = 0.3$$

Prinzipallösungen L-Welle

Nur Elektronen und einfach geladene Ionen:

$$n_L^2 = L = 1 - \frac{\omega_{pi}^2}{\omega(\omega - \omega_{ci})} - \frac{\omega_{pe}^2}{\omega(\omega + \omega_{ce})}$$

Resonanz: $n^2 = L = \infty \Rightarrow \omega = \omega_{ci}$

Cutoff: $n^2 = L = 0:$

$$\omega_L = \frac{\omega_{ci} - \omega_{ce}}{2} + \left[\left(\frac{\omega_{ci} + \omega_{ce}}{2}\right)^2 + \omega_p^2\right]^{\frac{1}{2}}$$

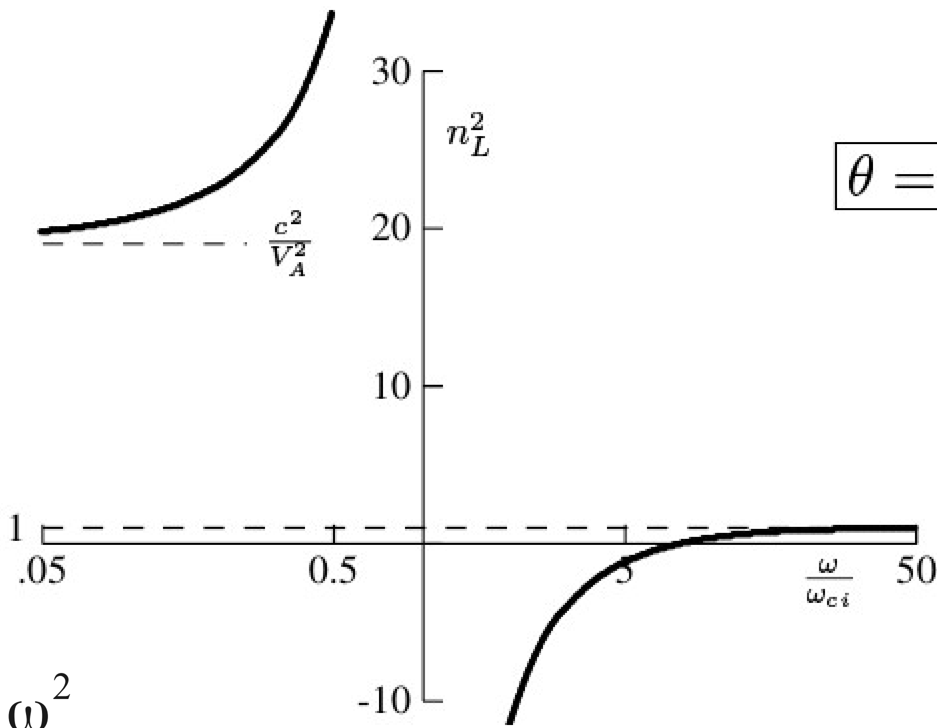
Mit: $\omega_p^2 = \omega_{pe}^2 + \omega_{pi}^2$

Näherungsweise, mit $m_e \ll m_i$: $\omega_L = \begin{cases} \omega_{ci} + \dfrac{\omega_{pi}^2}{\omega_{ci}} & \text{Cutoff, niedrige Dichte} \\ \omega_{pe} - \dfrac{1}{2}\omega_{ce} & \text{Cutoff, hohe Dichte} \end{cases}$

Prinzipallösungen L-Welle

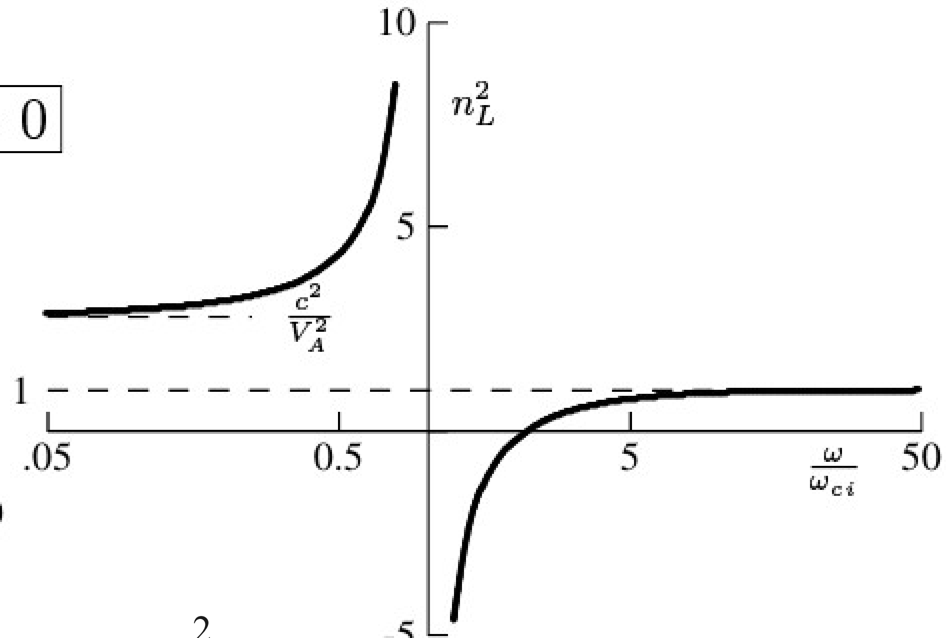
Für hohe / niedrige Frequenzen

$$n_L^2 = \begin{cases} 1 + \dfrac{\omega_{pi}^2}{\omega_{ci}^2} := \dfrac{c^2}{v_A^2} & \omega \to 0 \\ 1 - \dfrac{\omega_{pe}^2}{\omega^2} & \omega \to \infty \end{cases}$$



$$\frac{\omega_{pe}^2}{\omega_{ce}^2} = 3$$

$\theta = 0$



$$\frac{\omega_{pe}^2}{\omega_{ce}^2} = 0.3$$

Prinzipallösungen O-Welle

Ausbreitung senkrecht zu $B_0 \Leftrightarrow \theta = \frac{\pi}{2}$

$$\begin{vmatrix} S & -iD & 0 \\ iD & S-n^2 & 0 \\ 0 & 0 & P-n^2 \end{vmatrix} = 0 = S\left(S-n^2\right)\left(P-n^2\right) - D^2\left(P-n^2\right)$$

$$n_o^2 = P = 1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{pi}^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}$$

Wie im unmagnetisierten Fall.

Grund: das Elektrische Feld schwingt parallel zum Magnetfeld, mit dem Wellenfeld schwingende Teilchen „sehen“ das Magnetfeld daher nicht.

Prinzipallösungen X-Welle

Ausbreitung senkrecht zu $B_0 \Leftrightarrow \theta = \frac{\pi}{2}$

$$S(S-n^2)-D^2=0 \Rightarrow n^2 = S - \frac{D^2}{S} = \frac{S^2-D^2}{S}$$

$$n_x^2 = \frac{RL}{S} = \frac{[(\omega+\omega_{ci})(\omega-\omega_{ce})-\omega_p^2][(\omega-\omega_{ci})(\omega+\omega_{ce})-\omega_p^2]}{(\omega^2-\omega_{ci}^2)(\omega^2-\omega_{ce}^2)+\omega_p^2(\omega_{ce}\omega_{ci}-\omega^2)}$$

Resonanzen: Nullstellen des Nenners:

$$\omega^2 = \frac{\omega_e^2+\omega_i^2}{2} \pm \left[\frac{\omega_e^2-\omega_i^2}{2} + \omega_{pe}^2\omega_{pi}^2\right]^{\frac{1}{2}}$$

Mit: $\omega_j^2 = \omega_{cj}^2 + \omega_{pj}^2,\ j = e, i$

Prinzipallösungen X-Welle

Mit: $m_e \ll m_i$:

$$\omega^2 = \omega_{UH}^2 = \omega_{pe}^2 + \omega_{ce}^2$$ Upper-hybrid-resonance

Und: $$\omega^2 = \omega_{LH}^2 = \omega_{ce}\,\omega_{ci}\left(\frac{\omega_{pe}^2 + \omega_{ce}\,\omega_{ci}}{\omega_{pe}^2 + \omega_{ce}^2}\right)$$ Lower-hybrid-resonance

Cutoffs: Nullstellen des Zählers:

$$\omega^2 = \left[\left(\frac{\omega_{ce} + \omega_{ci}}{2}\right)^2 + \omega_p^2\right]^{\frac{1}{2}} \pm \frac{\omega_{ce} - \omega_{ci}}{2}$$

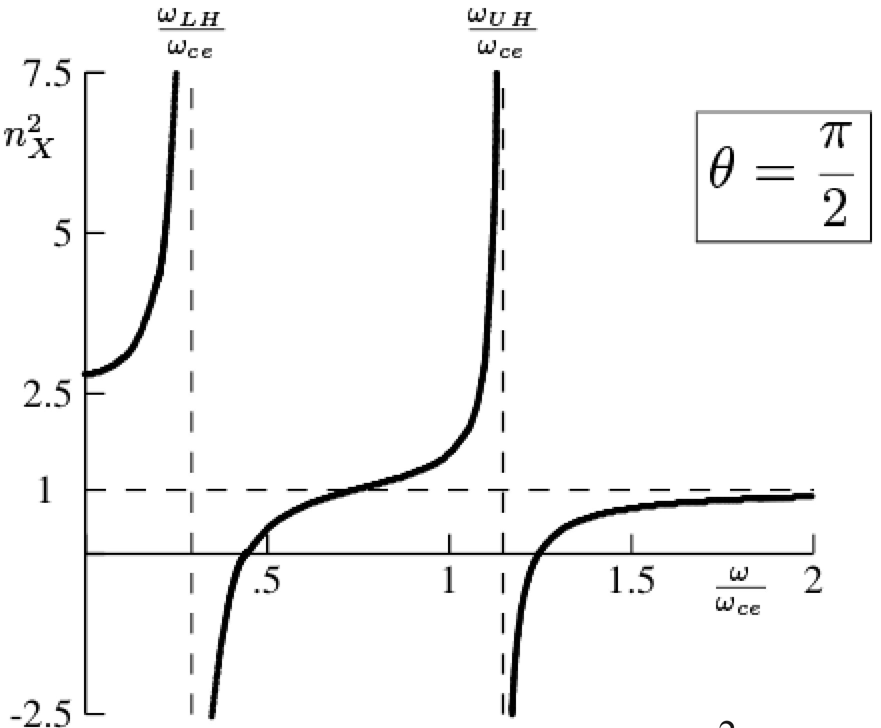
X-Wave-Cutoff, hohe Dichte

$$\omega_X \simeq \omega_{pe} \pm \frac{1}{2}\omega_{ce}$$

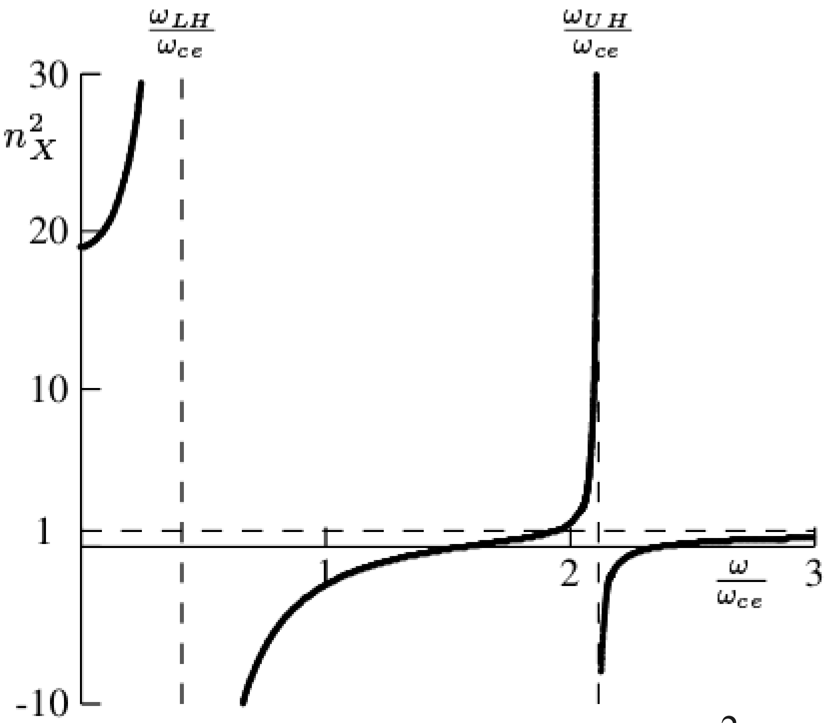
X-Wave-Cutoff, niedrige Dichte

$$\omega_X \simeq \begin{cases} \omega_{ce} + \dfrac{\omega_{pe}^2}{\omega_{ce}} \\ \omega_{ce} + \dfrac{\omega_{ce}^2}{\omega_{ce}} \end{cases}$$

Prinzipallösungen X-Welle



$$\frac{\omega_{pe}^2}{\omega_{ce}^2} = 0.3$$

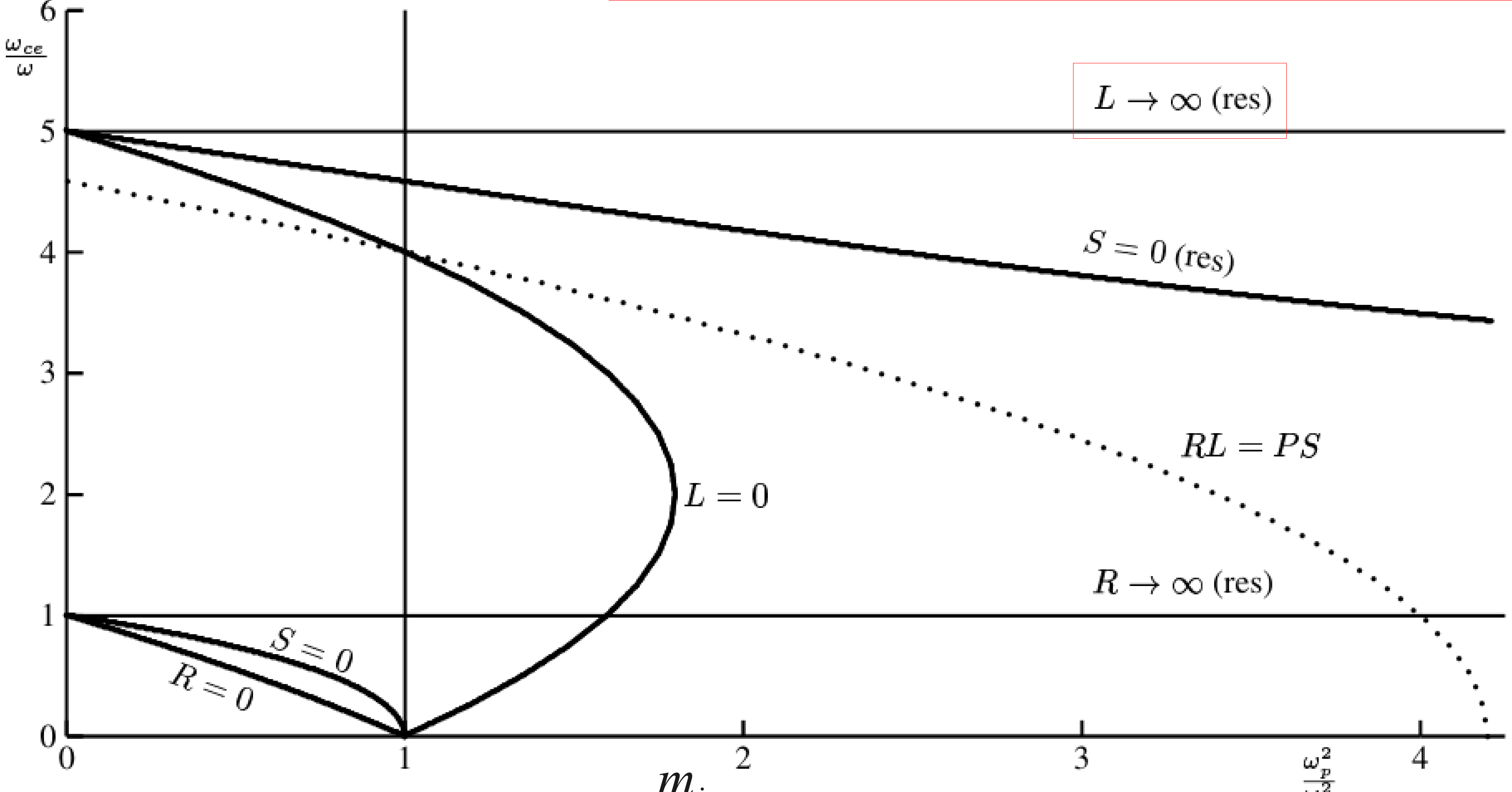


$$\frac{\omega_{pe}^2}{\omega_{ce}^2} = 3$$

CMA-Diagramm

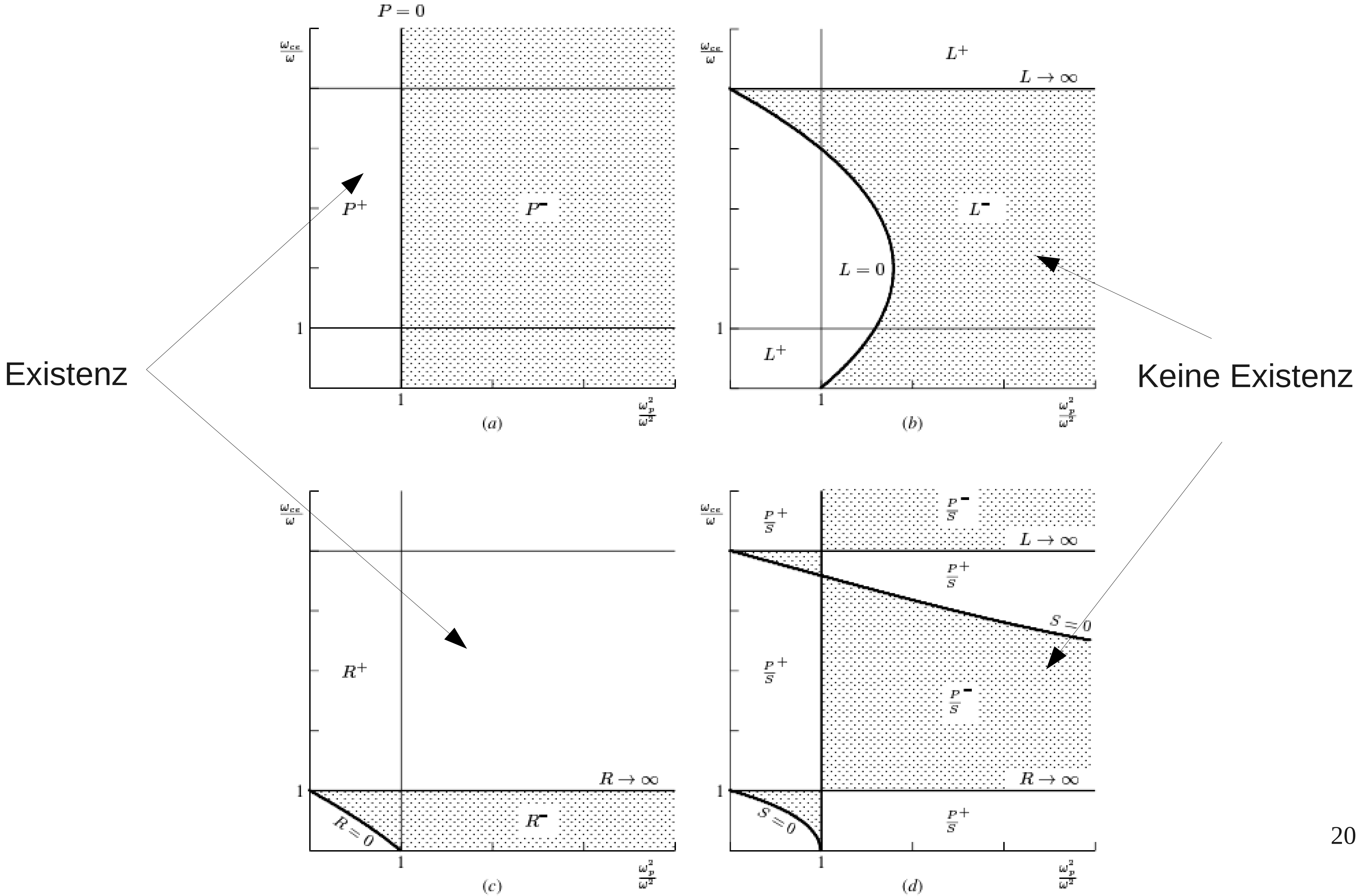
Cutoffs und Resonanzen als Funktion von Magnetfeld und Dichte, nach Clemmow, Mullaly und Allis.

$$L = 1 - \frac{\omega_{pi}^2}{\omega^2 - \omega\,\omega_{ci}} - \frac{\omega_{pe}^2}{\omega^2 + \omega\,\omega_{ce}} \rightarrow \infty \Rightarrow \omega^2 - \omega\,\omega_{ci} = 0 \Rightarrow \frac{\omega_{ci}}{\omega} = 1$$

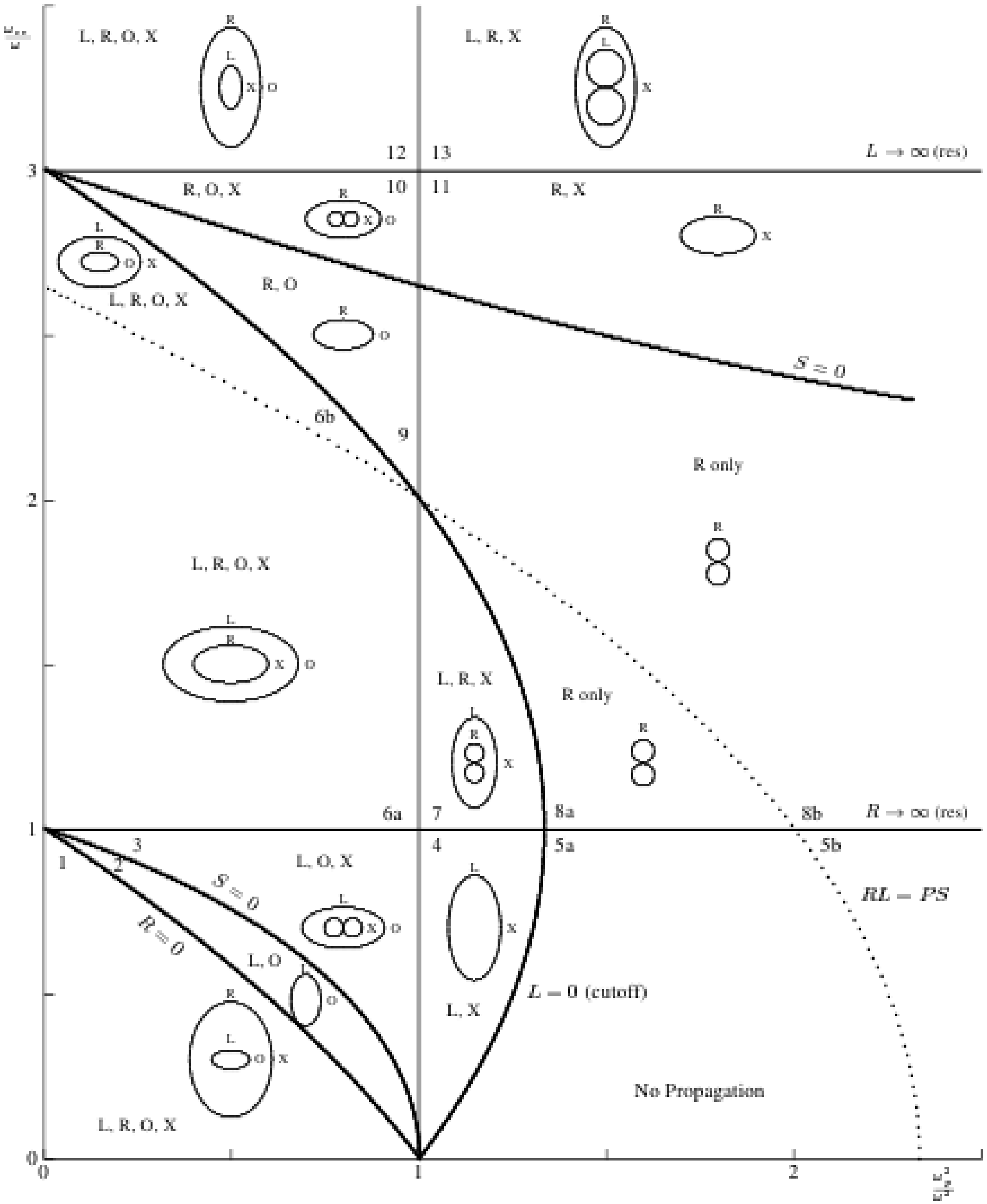


$$\frac{m_i}{m_e} = 5$$

CMA-Diagramm



CMA-Diagramm



$$\frac{m_i}{m_e} = 3$$

Gruppengeschwindigkeit (3D)

Wellenpaket: $$f(\vec{r},t)=\int A(\vec{k})\,e^{i(\vec{k}\cdot\vec{r}-\omega t)}\frac{d^3k}{(2\pi)^3}$$

Verteilt um $\vec{k}_0$, so dass $$\left|A(\vec{k})\right|\rightarrow 0,\ f\ddot{u}r\left|\vec{k}-\vec{k}_0\right|=\left|\Delta\vec{k}\right|\rightarrow\infty$$

t=0: $$f(\vec{r},0)=e^{i(\vec{k}_0\cdot\vec{r})}\int A(\vec{k})\,e^{i\Delta\vec{k}\cdot\vec{r}}\frac{d^3k}{(2\pi)^3}=e^{i\vec{k}_0\cdot\vec{r}}F(\vec{r})$$

Entwickeln von $\omega(\vec{k})$:

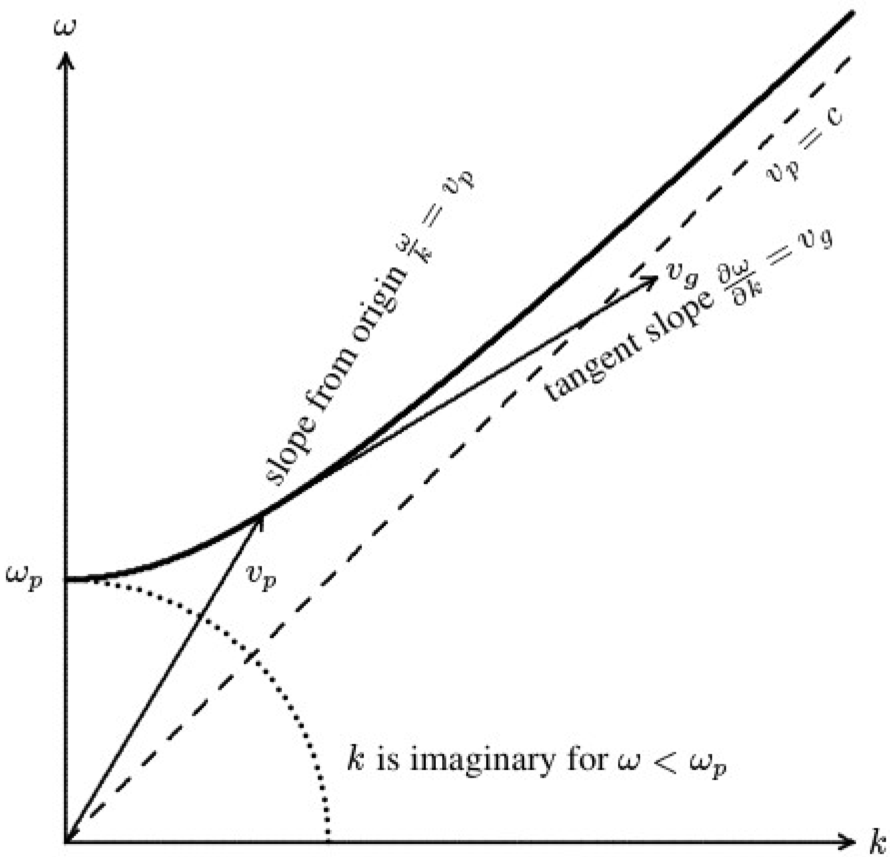
$$\omega(\vec{k})=\omega(\vec{k}_0)+\Delta\vec{k}\cdot\nabla_k\omega(\vec{k})_{k_0}+O(k^2)$$

Zeitliche Entwicklung:

$$f(\vec{r},t)=e^{i(\vec{k}_0\cdot\vec{r}-\omega t)}\int A(\vec{k})\exp\left[\Delta k\cdot\left(\vec{r}-\vec{\nabla}_k\omega(\vec{k})_{k_0}t\right)\right]\frac{d^3k}{(2\pi)^3}$$

Es folgt: $v_{ph}=\dfrac{\omega}{\left|\vec{k}_0\right|}$ und: $v_{gr}=\nabla_k\omega(\vec{k})_{k_0}$

Gruppengeschwindigkeit (3D)



Gruppengeschwindigkeit (3D)

$$\omega=\omega(k,\theta) \qquad \rightarrow \vec{v}_g=\left.\frac{\partial\omega}{\partial k}\right|_{\theta}\hat{k}+\frac{1}{k}\left.\frac{\partial\omega}{\partial\theta}\right|_{k}\hat{\theta}$$

Winkel zwischen $\vec{k}$ und $\vec{v}_{gr}$: $\tan(\alpha)=\frac{\theta\ Komponente}{k\ Komponente}=-\frac{1}{k}\left.\frac{\partial k}{\partial\theta}\right|_{\omega}$