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Enhanced Assumed Strain Elements and Nearly Incompressible Material

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Abstract. *Finite elements with enhanced assumed strains are now popular in the treatment of problems for which classical elements show a tendency of locking. The enhanced strains yield a softening of the energy functional. The basic tool for the analysis is a strengthened Cauchy inequality that is more easily handled than the Brezzi condition of the equivalent Hellinger–Reissner concept. It helps to eliminate checkerboard modes which have been observed in some computations, when nearly incompressible material is involved.*

1 Introduction

The elements with Enhanced Assumed Strains which were introduced by Simo and Rifai [7] are now popular in certain areas. Some computations with these EAS elements have turned out to show no locking in cases in which the corresponding non-enhanced elements suffer from locking.

Andelfinger and Ramm [1] observed already in numerical computations that the EAS concept in some model situation is equivalent to a Hellinger–Reissner formulation. Later Yeo and Lee [9] have given a formal proof which is based on the stiffness matrices of the variational formulation. We will provide a short proof that uses the variational formulation directly. It shows that the EAS method leads to a projection, and in this way to a softening of the functional for the stored energy. When the Hellinger–Reissner principle is used, the target space of the projector is specified by the user and by the finite element space for the stresses while the orthogonal complement is fixed with the EAS elements.

The equivalence above is verified by purely algebraic manipulations. Now it is well known that an LBB condition (due to Babuška, Brezzi, and Ladyšenskaya) is crucial for having stability of the mixed method derived from the Hellinger–Reissner principle. It follows from the equivalence that there will be a corresponding condition for the finite element spaces in the EAS concept.

Indeed, the stability is controled here by a *strengthened Cauchy inequality*. Such an inequality was discussed already by [2] and [5]. The inequality is easily verified for the popular EAS element proposed by [7].

Since the EAS method has been introduced mainly in order to suppress locking phenomena with finite elements of low degree, we will discuss the application to nearly incompressible material. We will show that volume locking is indeed not present provided that the space of enhanced strains is slightly enlarged. In this way we can also guarantee that checkerboard modes do not occur.

Specifically, we obtain convergence of the strains in a robust way, i.e., independently of the size of the Poisson ratio. The analysis does not include the volume term of the stress, and here we have a difference to mixed methods like the u-p-method. Nevertheless, the EAS method is a promising method with the advantage that we get discretizations with positive definite stiffness matrices.

2 The EAS-Method and it Softening Behaviour

Originally, Rifai and Simo [7] have introduced the EAS elements by starting from a formulation with the Hu–Washizu principle. Since the stability was not regarded, we prefer a more direct approach.

Let $\boldsymbol{v}$ denote the displacements of a body due to an external load. The Lamé equations for Hooke's material lead to the variational problem

$$\begin{aligned}\Pi(\boldsymbol{v}) &= \mu \int_\Omega \nabla^s \boldsymbol{v} : \nabla^s \boldsymbol{v}\, dx + \tfrac{\lambda}{2}\int_\Omega (\operatorname{div} \boldsymbol{v})^2 dx - \int_\Omega \boldsymbol{f}\boldsymbol{v}\, dx \\ &\to \text{min!}\end{aligned} \tag{1}$$

Here, the strains are given by the symmetric gradient

$$(\nabla^s \boldsymbol{v})_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right). \tag{2}$$

The quadratic part depends only on the displacement gradient. The solution $\boldsymbol{u}$ is characterized by the weak equation

$$(\mathcal{C}\nabla^s \boldsymbol{u}, \nabla^s \boldsymbol{v})_0 = (\boldsymbol{f}, \boldsymbol{v})_0 \quad \text{for all } \boldsymbol{v}. \tag{5}$$

With the elasticity tensor $\mathcal{C}$ being defined by

$$\boldsymbol{\sigma} = \mathcal{C}\boldsymbol{\epsilon} := 2\mu\boldsymbol{\epsilon} + \lambda(tr\boldsymbol{\epsilon})\mathbf{1} \tag{3}$$

and the usual L_2-inner product $(\cdot,\cdot)_0$ we have the short version of (1)

$$\Pi(\boldsymbol{v}) = \frac{1}{2}(\mathcal{C}\boldsymbol{v}, \boldsymbol{v})_0 - (\boldsymbol{f}, \boldsymbol{v}). \tag{4}$$

In the EAS method the trial functions for the displacements $\boldsymbol{v}_h$ are chosen from a finite element space V_h. The strains are now enriched by a finite element space $\tilde{E}_h$ of *enhanced assumed strains* that cannot be identified with symmetric gradients

$$\boldsymbol{\epsilon}_h = \nabla^s \boldsymbol{v}_h + \boldsymbol{\gamma}_h, \quad \boldsymbol{v}_h \in V_h, \boldsymbol{\gamma}_h \in \tilde{E}_h. \tag{6}$$

The enhacement makes the quadratic form less stiff since we have additional degress of freedom

$$\Pi(\boldsymbol{v}_h, \boldsymbol{\gamma}_h) = \frac{1}{2}(\mathcal{C}(\nabla^s \boldsymbol{v}_h + \boldsymbol{\gamma}_h),\ \nabla^s \boldsymbol{v}_h + \boldsymbol{\gamma}_h)_0 - (\boldsymbol{f}, \boldsymbol{v}_h)_0 \tag{7}$$

and the corresponding weak equations are:

Find $\boldsymbol{u}_h \in V_h$ and $\tilde{\boldsymbol{\epsilon}}_h \in \tilde{E}_h$ such that

$$\begin{aligned}(\mathcal{C}(\nabla^s \boldsymbol{u}_h + \tilde{\boldsymbol{\epsilon}}_h), \nabla^s \boldsymbol{v})_0 &= (\boldsymbol{f}, \boldsymbol{v})_0 && \forall \boldsymbol{v} \in V_h, \\ (\mathcal{C}(\nabla^s \boldsymbol{u}_h + \tilde{\boldsymbol{\epsilon}}_h), \gamma)_0 &= 0 && \forall \gamma \in \tilde{E}_h.\end{aligned} \tag{8}$$

2.1 The Equivalent Mixed Method and the Strengthened Cauchy Inequality

The resulting stresses $\boldsymbol{\sigma}_h = \mathcal{C}(\nabla^s \boldsymbol{u}_h + \tilde{\boldsymbol{\epsilon}}_h)$ are contained in a space S_h

$$\begin{aligned}S_h &= \mathcal{C}(\nabla^s V_h + \tilde{E}_h) \cap \tilde{E}_h^\perp \\ &= \{\boldsymbol{\tau}_h = \mathcal{C}(\nabla^s \boldsymbol{v}_h + \boldsymbol{\gamma}_h);\ \boldsymbol{v}_h \in V_h, \boldsymbol{\gamma}_h \in \tilde{E}_h, \text{ and } (\boldsymbol{\tau}_h, \tilde{\gamma})_0 = 0\ \forall \tilde{\gamma} \in \tilde{E}_h\}\end{aligned} \tag{9}$$

It is clear that the stresses are in the sum of the two finite element spaces as stated in (9). The orthogonality relation follows from the weak equation $(8)_2$.

Now the following result is obtained by simple algebraic manipulations; cf. [9, 4] or [3, Chapter III. 5].

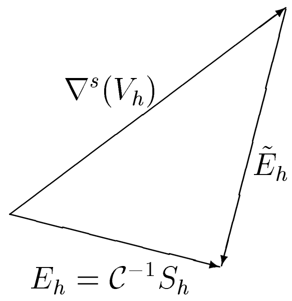


Figure 1. $\mathcal{C}$-orthogonal decomposition of $\nabla^s(V_h)$ shows the relationship between the spaces V_h, $\tilde{E}_h$, and S_h, i.e., $\mathcal{C}^{-1}\boldsymbol{\sigma}_h = \nabla^s \boldsymbol{v}_h + \tilde{\boldsymbol{\epsilon}}_h$.

Theorem. Let the finite element spaces $V_h, \tilde{E}_h$, and S_h be related by (9). Then the weak equations (8) for the EAS formulation are equivalent to the mixed method: *Find* $\boldsymbol{u}_h \in V_h$ *and* $\boldsymbol{\sigma}_h \in S_h$ *such that*

$$\begin{aligned} (\mathcal{C}^{-1}\boldsymbol{\sigma}_h, \boldsymbol{\tau})_0 - (\boldsymbol{\tau}, \nabla^s \boldsymbol{u}_h) &= 0 & \forall \boldsymbol{\sigma} \in S_h, \\ -(\boldsymbol{\sigma}_h, \nabla^s \boldsymbol{v})_0 &= -(\boldsymbol{f}, \boldsymbol{v}) & \forall \boldsymbol{v} \in V_h. \end{aligned}$$

In particular, $\boldsymbol{\sigma}_h = \mathcal{C}(\nabla^s \boldsymbol{u}_h + \tilde{\boldsymbol{\epsilon}}_h)$. Moreover $\mathcal{C}^{-1}\boldsymbol{\sigma}_h$ is the projection of $\nabla^s \boldsymbol{u}_h$ to the space $\mathcal{C}^{-1}S_h$ with respect to the energy norm.

Since such a projection reduces the internal energy (and does not magnify it), the softening character is easily understood.

2.2 The EAS4 Element

The most numerical investiations within the framework of EAS elements have been done for the 2-dimensional elasticity problem with the EAS4 element, the EAS7 element, and the PS5 element. Moreover, they are attractive because of the resemblance with Wilson's nonconforming element, see [8] and with the mixed method of Pian and Sumihara [6]. The three models coincide, if the domains Ω is a rectangle and can be decomposed into quadratic elements of mesh size h.

We consider a uniform mesh of square elements generated by affine maps from the reference square

$$T_{ref} = [-1, +1]^2 = \{(\xi, \eta) \in \mathbb{R}^2; -1 \le \xi, \eta \le +1\}.$$

The finite element space V_h contains Q_1 elements, i.e., bilinear elements for each component u_i of the displacement $\boldsymbol{u} = (u_1, u_2)$, and we have local coordinates (ξ, η):

$$u_i = a_i + b_i\xi + c_i\eta + d_i\xi\eta, \quad i = 1, 2. \tag{10}$$

In plane elasticity the strains are commonly written in the vector form $\boldsymbol{\epsilon} = (\epsilon_{11}, \epsilon_{22}, 2\epsilon_{12})$. Hence. the space of consistent strains $\nabla^s V_h$ is locally spanned by the columns of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & \eta & 0 \\ 0 & 1 & 0 & 0 & \xi \\ 0 & 0 & 1 & \xi & \eta \end{pmatrix}. \tag{11}$$

The finite element space $\tilde{E}_h$ is given by the matrix

$$\begin{pmatrix} \xi & 0 & 0 & 0 \\ 0 & \eta & 0 & 0 \\ 0 & 0 & \xi & \eta \end{pmatrix}. \tag{12}$$

The constant for a strengthened Cauchy inequality is easily calculated since most of the pairs are orthogonal. A nontrivial inner product occurs with the 4th column for $\nabla^s V_h$ and the 3rd one for $\tilde{E}_h$. Generally we have for square elements

$$(\nabla^s \boldsymbol{v}_h, \boldsymbol{\gamma}_h)_0 \le \frac{1}{\sqrt{2}} \|\nabla^s \boldsymbol{v}_h\|_0 \, \|\boldsymbol{\gamma}\|_0.$$

In the case of rectangles the number 2 has to be replaced by the quantity 1+mesh ratio.

Now, assume that we have a nearly incompressible material, i.e.,

$$\lambda \gg \mu.$$

Then we have obviously solutions with a small divergence, we expect that

$$\| \operatorname{div} \boldsymbol{u}\|_0 \ll |\boldsymbol{u}|_1.$$

This corresponds to the fact that there are enough vector fields $\boldsymbol{v}$ with $\operatorname{div} \boldsymbol{v} = 0$.

The situation is different for the discretization with the Q_1 element. The kernel of the divergence operator is only a 5-dimensional space. The functions $\boldsymbol{v}_h^{kern}$ in V_h with $\operatorname{div} \boldsymbol{v}_h^{kern} = 0$ are global polynomials von Ω (no jumps of derivatives at interelement faces)

$$\boldsymbol{v}_h^{kern} = \begin{pmatrix} a + bx + cy \\ b + cx - by \end{pmatrix}. \tag{13}$$

The divergence of the functions outside of the kernel can be bounded from below [2].

Theorem. Let Ω be a rectangle with diameter d, and V_h be the space of bilinear elements on meshes with squares of mesh size h. Then given $\boldsymbol{v}_h \in V_h$, there is a function $\boldsymbol{v}_h^{kern}$ of the form (13) such that

$$\| \operatorname{div} \boldsymbol{v}_h\|_{0,\Omega} \ge \frac{h}{12d} |\boldsymbol{v}_h - \boldsymbol{v}_h^{kern}|_{1,\Omega}. \tag{14}$$

It follows that the finite element solution locks unless the exact solution has the special form (13). It is also clear that the locking effect disappears for $h \to 0$, but in actual computations the mesh sizes are far away from this limit.

It is well known that (14) is no longer true if we replace

$$\int_\Omega (\operatorname{div} \boldsymbol{v})^2 dx$$

by a numerical integral with 1-point evaluation. Roughly speaking, the 1-point integration is modelled by this EAS element. Indeed, given $\boldsymbol{v}_h \in V_h$ we see that $\operatorname{div} \boldsymbol{v}_h$ is linear in each element, and we can choose $\boldsymbol{\gamma}_h \in \tilde{E}_h$ such that the effective divergence

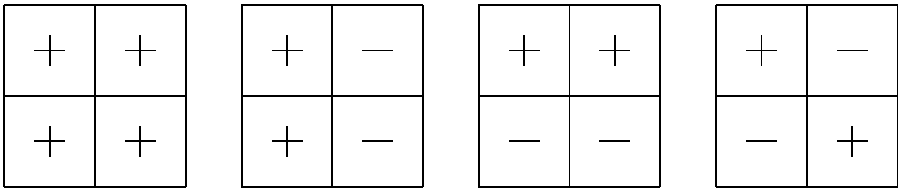
$$\operatorname{div} \boldsymbol{v}_h + trace\, \boldsymbol{\gamma}_h \tag{15}$$

is a piecewise constant function.

2.3 Final Modification

The locking effect is reduced substantially in this way. Only a small rest remains in the form of a tendency to create checkerboard modes for the volume term. It is related to the checkerboard instability of the $Q_1 - P_0$-element for the Stokes problem.

We will eliminate this rest of drawback by making that the effective divergence (15) is *piecewise constant on macroelements.* To this end we include the piecewise constant functions shown in Fig. 2b-d into the set of enhanced assumed strains. In this way we obtain; cf. [4].



(a) (b) (c) (d)

Figure 2 a-d. Basis functions of $\tilde{E}_h$ on the macro-elements with values ± 1 according to the sign indicated inside the elements.

Theorem. There is a constant c independent of h and the Poisson ratio λ such that

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_1 + \|\tilde{\boldsymbol{\epsilon}}_h\|_0 \le ch\|\boldsymbol{f}\|_0.$$

Here it is even possible to establish a posteriori error estimators for getting optimal meshes by adaptive mesh refinements. For details see [4].

3 Conclusions

Enhanced assumed strains are an appropriate tool for the suppression of locking phenomena with finite elements of low degree. Specifically, the popular elements proposed by Simo and Rifai substantially reduce volume locking. The remaining disadvantage of possible checkerboard modes can be eliminated by further enhanced strains on macroelements. The equivalence with suitable Hellinger–Reissner formulation shows that a strengthened Cauchy inequality is related to stability of the element.

The implementation of the EAS leads to stiffness matrices that are positive definite. This is an adavantage when compared with the mixed methods (as the u-p-method) although the latter are more favorable in suppressing locking from a theoretical viewpoint.

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