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CONSTRUCTIVE
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Note on the Approximation of Powers of the Distance in Two-Dimensional Domains

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Abstract. Although Newman's trick has been mainly applied to the approximation of univariate functions, it is also appropriate for the approximation of multivariate functions that are encountered in connection with Green's functions for elliptic differential equations. The asymptotics of the real-valued function on a ball in 2-space coincides with that for an approximation problem in the complex plane. The note contains an open problem.

In the numerical treatment of elliptic equations there arises the problem in [4] of polynomial approximation of

$$f(x, y) := [(x - x_0)^2 + (y - y_0)^2]^{-s} \tag{1}$$

since the fundamental solutions often contain terms of this form. Here $s > 0$. Although this is a multivariate approximation problem with two real variables, it can be reduced to an approximation problem in the complex plane by a simple construction which goes back to Donald Newman [3] and is called Newman's trick. In this way we also obtain a sharp lower bound of the error of the best approximation, and the exponential decrease of the error is easily established.

We are interested in the approximation of f on a ball in 2-space that does not contain the point (x_0, y_0). After a translation and scaling, if necessary, we may assume that the ball is the unit ball. Moreover, we may apply a rotation such that (x_0, y_0) is mapped to the point $(r, 0)$ on the real line. By assumption, $r > 1$.

As usual, we define

$$E_n(f) := \inf\{\| f - P_n\|;\ \ P_n \text{ is a polynomial of total degree } n\},$$

where $\|\cdot\|$ refers to the sup-norm on the unit ball. Our aim is an estimate of the asymptotic behavior.

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Theorem. *Assume that* $r^2 := x_0^2 + y_0^2 > 1$. *Then we have for the polynomial approximation on the unit disk*

$$\overline{\lim_{n\to\infty}} E_n(f)^{1/n} \le \frac{1}{r}. \tag{2}$$

If, moreover, $r \ge 3$ *or* $0 < s < 1$, *then*

$$\overline{\lim_{n\to\infty}} E_n(f)^{1/n} = \frac{1}{r}. \tag{3}$$

It is an open problem whether the lower bound also holds if r gets close to 1 and if s is large. On the other hand, the estimate (3) can be extended to the function $\log[(x - x_0)^2 + (y - y_0)^2]$ by considering it as the real part of an analytic function.

1. Adaptation of Newman's Construction

First we turn to the complex-valued function

$$g(z) := (r - z)^{-s}.$$

Obviously,

$$|g(z)| \le k^s (r-1)^{-s} \qquad \text{for} \quad |z| \le r - \frac{r-1}{k}. \tag{1}$$

Let p_k be the Taylor polynomial of degree k for g. Cauchy's integral formula yields the representation of the remainder

$$g(z) - p_k(z) = z^{k+1} \frac{1}{2\pi i} \oint_{|z|=r-\varepsilon} \frac{1}{t-z} \frac{1}{t^{k+1}} g(t)\, dt. \tag{2}$$

Following Bernstein's well-known arguments for the approximation of analytic functions, see, e.g., [2, Chap. 7, §8] we obtain

$$|g(z) - p_k(z)| \le c_1 k^s r^{-k} \qquad \text{for} \quad |z| \le 1, \tag{3}$$

where $c_1 \le e(r-1)^{-s-1}$. The factor e enters here through the formula $(1-1/k)^{-k} \le e$.

The polynomials p_k are not real valued, but the product

$$P_{2n}(x,y) := p_n(z)\overline{p_n(z)}$$

is a real polynomial of degree $2n$ in the real variables x and y. This is easily verified by looking at the product of linear factors

$$\begin{aligned}(z-a)(\bar z - \bar a) &= x^2 + y^2 + |a|^2 - 2\,\Re(\bar a z) \\ &= x^2 + y^2 + |a|^2 - 2x\,\Re a - 2y\,\Im a.\end{aligned} \tag{4}$$

Next we note that

$$g(z)\overline{g(z)} = |z - r|^{-2s}.$$

Now, Newman's trick [3] applies to the product; see also [1, p. 139]:

$$g\bar{g} - p_n\bar{p}_n = 2\,\Re[\bar{g}(g - p_n)] - |g - p_n|^2. \tag{5}$$

Hence,

$$\begin{aligned} |g\bar{g} - p_n\bar{p}_n| &\le 2(r-1)^{-s}c_1 n^s r^{-n} + c_1^2 n^{2s} r^{-2n} \\ &\le c_2 n^{2s} r^{-n} \qquad \text{for} \quad |z| \le 1. \end{aligned} \tag{6}$$

Here, $c_2 := 2c_1(r-1)^{-s} + c_1^2$. The bound is conservative since the second term in (9) is dominated by the first one if n is large. After inserting the real-valued functions we obtain

$$|[(x-r)^2 + y^2]^{-s} - P_{2n}(x, y)| \le c_2 n^{2s} r^{-n} \qquad \text{for} \quad x^2 + y^2 \le 1. \tag{7}$$

2. An Improvement

In a second step we will construct a polynomial of total degree n which approximates f of the same order as P_{2n}. Let p_k be as above and define

$$\begin{aligned} q_0 &:= p_0, \\ q_k &:= p_k - p_{k-1}, \qquad k \ge 1. \end{aligned}$$

For improving the approximation we set

$$Q_n(x, y) := \sum_{\substack{k,\ell=0 \\ (k+\ell \le n)}}^{n} q_k(z)\overline{q_\ell(z)}. \tag{1}$$

Since the symmetrical product $q_k(z)\overline{q_\ell(z)} + q_\ell(z)\overline{q_k(z)}$ is real valued, so is the sum in (11). Obviously, we have

$$P_{2n}(x, y) - Q_n(x, y) = \sum_{\substack{k,\ell=0 \\ (k+\ell > n)}}^{n} \overline{q_k(z)}q_\ell(z) = \sum_{k=1}^{n} \overline{q_k(z)}\,[p_n(z) - p_{n-k}(z)].$$

From (6) it follows that

$$|p_\ell - p_k| \le |g - p_k| + |g - p_\ell| \le 2c_1 \ell^s r^{-k} \qquad \text{if} \quad k < \ell.$$

Moreover, when estimating the derivative $g^{(k)}(0)$ via Cauchy's integral theorem, we recall (4) and see that $|q_k(z)| \le c_1 k^s r^{-k} |z|^k$. Hence,

$$\begin{aligned} |P_{2n}(x, y) - Q_n(x, y)| &\le \sum_{k=1}^{n} c_1 n^s r^{-k}\, 2c_1 n^s r^{-n+k} \\ &= 2c_1^2 n^{2s+1} r^{-n}. \end{aligned} \tag{2}$$

Set $c_3 := 2c_1^2 + c_2$. Combining (10) and (12) we obtain

$$E_n(f) \le c_3 n^{2s+1} r^{-n}. \tag{3}$$

This proves the upper bound for the asymptotics as stated in (2).

3. The Lower Bound

The advantage of the construction via an approximation in the complex plane becomes apparent when we establish the lower bound. The latter refers to the fact that the approximation on the unit ball cannot be better than the approximation on the unit circle.

It follows from (7) that $p_n(z)\overline{p_n(z)}$ is merely a polynomial of degree n if its restriction to the unit circle is considered; see [3]. Let $\tilde{P}_n := P_{2n}|_{|z|=1}$. Now the approximation problem on the unit circle is equivalent to an approximation problem with trigonometric polynomials. The winding number argument in [1, pp. 139 and 147] or [5] shows that $p_n\bar{p}_n$ provides a good test function for applying de la Vallée-Poussin's theorem.

To be specific, we have

$$\arg\{\bar{g}(g - p_n)\} = \arg\{g^{-1}(g - p_n)\}.$$

Since $g - p_n$ has a zero of multiplicity $n + 1$, the winding number of $g^{-1}(g - p_n)$ is $n + 1$, and the argument of $\bar{g}(g - p_n)$ is increased by $2\pi(n + 1)$ when the unit circle is transversed. Therefore $\Re\{\bar{g}(g - p_n)\} = \bar{g}(g - p_n)$ holds at least at $2n + 2$ points with alternating signs, and from de la Vallée-Poussin's theorem and (8) we conclude that for any polynomial P_n of degree n:

$$\| f - P_n\| \geq 2 \min_{|z|=1}\{|g(g - p_n)|\} - \mathcal{O}(n^{2s}r^{-2n}). \tag{1}$$

The crucial point is that we will get good lower estimates of the first term on the right-hand side in the case

$$r \geq 3 \qquad \text{or} \qquad 0 < s < 1. \tag{2}$$

If $r \geq 3$, a near circularity property in the sense of [5] is easily established. Assume that n is so large that $(n + 1)^s < \frac{6}{5}n^s$. Then we have $|q_{k+1}(z)| \leq \frac{2}{5}|q_k(z)|$ if $k > n$ and $|z| = 1$. A summation of the geometric series yields $\sum_{k\geq n+2} |q_k(z)| \leq \frac{2}{3}|q_{n+1}(z)|$; i.e., the next term of the power series dominates the remainder,

$$|g(z) - p_n(z)| \geq \tfrac{1}{3}|q_{n+1}(z)| \qquad \text{for} \quad |z| = 1. \tag{3}$$

For determining q_{n+1} we use here the explicit representation of $g^{(n+1)}$:

$$\frac{g^{(n+1)}(0)}{(n + 1)!} = \frac{s(s + 1)\cdots(s + n)}{(n + 1)!}r^{-s-n-1} \geq \frac{s}{n + 1}r^{-s-n-1}.$$

Combining the last two estimates with an obvious lower bound of $|g|$ we obtain

$$|g(z)\,[g(z) - p_n(z)]| \geq \tfrac{1}{3}(r + 1)^{-s}\frac{s}{n + 1}r^{-s-n-1} \qquad \text{for} \quad |z| = 1.$$

Now, we apply (14), and in this way de la Vallée-Poussin's theorem asserts that

$$E_n(f) \geq c_4 n^{-1}r^{-n-1} - \mathcal{O}(n^{2s}r^{-2n}) \qquad \text{for } n \text{ sufficiently large} \tag{4}$$

with $c_4 > 0$. The lower bound in (3) has now been verified for the case $r \geq 3$.

Obviously, the same technique can be used more generally for $r > 2$ if the constants $\frac{6}{5}$ and $\frac{1}{3}$ are replaced by appropriate r-dependent factors.

Another case in which we can establish a good de la Vallée-Poussin bound refers to $0 < s < 1$. Here the Cauchy integral (5) can be deformed into an integral along the cut (r, ∞) on the real line

$$g(z) - p_n(z) = z^{n+1} \frac{1}{\pi} \int_r^\infty \frac{1}{t-z} \frac{1}{t^{n+1}} (\Im g(t))\, dt.$$

The weight factor $t^{-n-1} \Im g(t)$ in the integral is positive, and we obtain for $|z| = 1$:

$$\begin{aligned} |g(z) - p_n(z)| &\geq \frac{1}{\pi} \int_r^\infty \Re \frac{1}{t-z} \frac{1}{t^{n+1}} (\Im g(t))\, dt \\ &\geq \frac{r}{r+1} \frac{1}{\pi} \int_r^\infty \frac{1}{t^{n+2}} (\Im g(t))\, dt \\ &= \frac{r}{r+1} \frac{1}{n+1!} g^{(n+1)}(0). \end{aligned}$$

From $r/(r+1) \geq \frac{1}{3}$ it follows that (16) holds here also, and we obtain again the lower bound in (17). This completes the proof of the theorem.

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