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A cascadic multigrid algorithm for the Stokes equations

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Summary. A variant of multigrid schemes for the Stokes problem is discussed. In particular, we propose and analyse a cascadic version for the Stokes problem. The analysis of the transfer between the grids requires special care in order to establish that the complexity is the same as that for classical multigrid algorithms.

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1. Introduction

Multilevel methods *without* coarse grid corrections have been defined and applied to elliptic problems of second order by Bornemann and Deuflhard [1,5]. They have called it a cascadic algorithm and showed that an optimal iteration method with respect to the energy norm is obtained if conforming elements are used.

Deuflhard's starting point for the cascadic multigrid method [5] was the idea that it should be sufficient to start the iteration at the level i with a good approximation from the level $i-1$. A similar idea can already be found in Chapter 9 of Wachspress' book [12] from 1966, i.e. from the period in which also the first theoretical investigations of multigrid methods were made. The approach from that period had, however, the drawback that not enough steps were performed on the coarse grids. Later Shaidurov [8] established in essence a recursion relation of the form

$$\|u_i - v_i\|_1 \le \|u_{i-1} - v_{i-1}\|_1 + c\frac{h_i}{m_i} \tag{1.1}$$

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for some finite element problems with full regularity. Here u_i denotes the exact solution on the level i and v_i its approximation computed after m_i steps. The accumulation of the error is no problem since the iteration steps on the lower levels are cheap.

It is crucial for the optimality of the algorithm that the error from the previous level enters with a factor of precisely 1. Since it was not clear whether a constant factor *greater than* 1 is encountered in the transfer for nonconforming elements, there are no serious conjectures for the latter families. This feature is shared by the Stokes equations as will be obvious in Sect. 4. In fact, the nonconformity caused by the prolongation operator introduces factors strictly greater than 1 in the recursion (1.1).

There is another difference to classical multigrid algorithms. The recursion relation (1.1) refers only to the energy norm, and it has been proved in [2] that the cascadic version is in general *not* optimal for the L_2-norm. This is in contrast to classical multigrid algorithms, see [7, 13], where one can more easily move between the H^1-norm and the L_2-norm.

We will develop the cascadic multigrid method for saddle point problems which arise from the Stokes problem. Here we will apply the smoothing procedure proposed in [4]. However, prolongating an approximate solution to the next higher level generally destroys the divergence freeness ensured by the smoother. Since the natural correction arising in this context involves a projector that is orthogonal in L_2 and not with respect to the energy inner product, there is the drawback with the L_2-norm mentioned above. Nevertheless, we are able to properly isolate the influence of nonconformity and to apply then a duality technique providing sufficiently sharp estimates for the additional terms. This eventually will be shown to yield optimality for our saddle point problems. Since this in turn is related to a careful analysis of the transfer between the grids, the technique is also useful in the treatment of nonconforming elements.[1] The analysis shows that the loss induced by the transfer between the grids can be controled also in nonstandard cases. This may be of interest for many multigrid algorithms and not only for those of cascadic type.

We note that the cascadic multigrid algorithm offers an efficient alternative to nested iteration for obtaining a good initial guess of the finite element solution. It is not our intention to replace the standard multigrid procedure.

2. Notation and problem formulation

Let $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, and let $H^s(\Omega)$, $H^s_0(\Omega)$ denote the usual Sobolev spaces endowed with the Sobolev norms $\|\cdot\|_s$. The space $L_{2,0}(\Omega) :=$

[1] *Note added in proof.* Indeed, R. Stevenson [9] told the authors in Jan. 1998 that he has applied the analogous technique to deal with nonconforming elements

$\{q \in L_2(\Omega) : \int_\Omega q dx = 0\}$ can be identified with $L_2(\Omega)/\mathbb{R}$. The weak formulation of the Stokes problem reads: Find $u \in X := H_0^1(\Omega)^d$ and $p \in M := L_{2,0}(\Omega)$ such that

$$\begin{aligned} a(u,v) + b(v,p) &= \langle f, v\rangle \text{ for all } v \in X, \\ b(u,q) &= 0 \qquad \text{for all } q \in M. \end{aligned} \tag{2.1}$$

Here, $f \in X'$, the dual of X, is given, with $\langle \cdot, \cdot \rangle$ being the standard duality pairing induced by the L_2 inner product, and

$$\begin{aligned} a(u,v) &:= \int_\Omega \nabla u \nabla v dx, \\ b(v,q) &:= -\int_\Omega \operatorname{div} v q dx. \end{aligned} \tag{2.2}$$

We assume that the problem is H^2-regular, e.g. Ω may be a bounded convex polyhedral domain in 2-space.

We are interested in approximate solutions to (2.1) obtained by finite element discretizations. To this end we assume that for each $i \in \mathbb{N}_0$, $i \le J$, T_i denotes a shape-regular triangulation of Ω which is generated by successively refining uniformly some initial triangulation T_0. Shape regularity means that the ratio of the diameter and the radius of the largest inscribed ball of any simplex in T_i remains bounded. Accordingly, X_i and M_i will denote the corresponding conforming finite element spaces of Taylor and Hood [3, 6]. Likewise we may use any elements with the properties listed in [11]. In particular, the finite element spaces are nested and form an ascending hierarchy of spaces

$$X_0 \subset X_1 \subset \cdots \subset X_J \subset X, \qquad M_0 \subset M_1 \subset \cdots \subset M_J \subset M.$$

Restricting (2.1) to the pair X_i, M_i, gives rise to the linear system of equations

$$\begin{pmatrix} A_i & B_i^{\mathrm{T}} \\ B_i & \end{pmatrix} \begin{pmatrix} u_i \\ q_i \end{pmatrix} = \begin{pmatrix} f_i \\ 0 \end{pmatrix}, \tag{2.3}$$

where as usual the operators A_i, B_i on X_i are for $u_i \in X_i$ defined by

$$(A_i u_i, v) = a(u_i, v), \quad v \in X_i, \qquad (B_i u_i, q) = b(u_i, q), \quad q \in M_i.$$

Of course, as soon as one fixes bases in X_i and M_i, one obtains matrix representations of A_i, B_i which will be denoted again by A_i, B_i, respectively. For simplicity we identify the functions v_i, q_i in X_i, M_i with their coefficient sequences, always assuming that the bases are normalized so that

$$\|v_i\|_0 \sim \|v_i\|_{\ell_2}. \tag{2.4}$$

That is, both norms can be uniformly bounded by constant multiples of each other. Moreover we have the inverse inequalities

$$\|v_i\|_1 \le c h_i^{-1} \|v_i\|_0, \quad v_i \in X_i. \tag{2.5}$$

Here and throughout the paper c will be a generic constant which is independent of the level and which may be different in different equations.

Our objective is to solve (2.3) for the highest level of resolution $i = J$.

3. The smoothing operation

A key ingredient of a multigrid scheme for the solution of (2.3) is a suitable smoother. In the following we will employ the smoother proposed in [4]. Since this can be described for an abstract saddle point problem, for convenience we suppress the subscripts indicating the discretization level. Thus we consider the linear system of equations

$$\begin{pmatrix} A & B^{\mathrm{T}} \\ B & \end{pmatrix} \begin{pmatrix} u \\ q \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}, \tag{3.1}$$

where A is a symmetric positive definite matrix. It characterizes the solution of the constrained minimum problem

$$\frac{1}{2} u^{\mathrm{T}} A u - f^{\mathrm{T}} u \to \text{min! subject to } Bu = g.$$

Now suppose that C is a preconditioner for A which, in particular, satisfies

$$v^{\mathrm{T}} A v \le v^{\mathrm{T}} C v, \quad v \in X, \tag{3.2}$$

and for which the linear system

$$\begin{pmatrix} C & B^{\mathrm{T}} \\ B & \end{pmatrix} \begin{pmatrix} v \\ q \end{pmatrix} = \begin{pmatrix} d \\ e \end{pmatrix}, \tag{3.3}$$

is more easily solved. Note that the inverse is formally given by

$$\begin{pmatrix} C & B^{\mathrm{T}} \\ B & \end{pmatrix}^{-1} = \begin{pmatrix} C^{-1}(I - B^{\mathrm{T}} S^{-1} B C^{-1}) & C^{-1} B^{\mathrm{T}} S^{-1} \\ S^{-1} B C^{-1} & -S^{-1} \end{pmatrix}, \tag{3.4}$$

where

$$S := B C^{-1} B^{\mathrm{T}}$$

is the Schur complement of (3.3). Specifically, if $C = \alpha I$, then (3.2) reads

$$v^{\mathrm{T}} A v \le \alpha\, v^{\mathrm{T}} v,$$

i.e., α is assumed to be not smaller than the spectral radius $\rho(A)$ of A. In this case (3.4) becomes

$$\begin{pmatrix} \alpha I & B^{\mathrm{T}} \\ B & \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{\alpha} P & B^{\mathrm{T}}(BB^{\mathrm{T}})^{-1} \\ (BB^{\mathrm{T}})^{-1}B & -\alpha(BB^{\mathrm{T}})^{-1} \end{pmatrix}, \tag{3.5}$$

where P is the projection

$$P := I - B^{\mathrm{T}}(BB^{\mathrm{T}})^{-1}B. \tag{3.6}$$

Now, (3.1) is to be solved by an iteration of the form

$$\begin{pmatrix} u^{\ell+1} \\ p^{\ell+1} \end{pmatrix} := \begin{pmatrix} u^{\ell} \\ p^{\ell} \end{pmatrix} - \begin{pmatrix} \alpha I & B^{\mathrm{T}} \\ B & \end{pmatrix}^{-1} \left\{ \begin{pmatrix} A & B^{\mathrm{T}} \\ B & \end{pmatrix} \begin{pmatrix} u^{\ell} \\ p^{\ell} \end{pmatrix} - \begin{pmatrix} f \\ g \end{pmatrix} \right\}, \tag{3.7}$$

where superscripts will always denote iteration indices. It is important to note that $u^{\ell+1}$ always satisfies the constraint, i.e.,

$$Bu^{\ell+1} = g, \tag{3.8}$$

see [4]. Each iteration step requires solving a system of the form (3.3) with $C = \alpha I$. By (3.5), this can be realized by implementing

$$BB^{\mathrm{T}}q = Bd - \alpha e, \quad v = \frac{1}{\alpha}(d - B^{\mathrm{T}}q).$$

Specifically, this amounts to solving an equation similar to the Poisson equation in the case of the Stokes problem. In view of the available efficient Poisson solvers this is acceptable, e.g., smoothers which incorporate Poisson solvers have been used in some efficient multigrid algorithms by Turek [10]. Moreover numerical results in [4] support the expectation that approximate solutions of the equations are sufficient. Obviously it would be against the spirit of the idea of cascadic solvers to use multigrid here, but fortunately there are efficient AMG algorithms which solve the Poisson equation in a black box manner.

In particular, defining for $g = 0$

$$V := \{v \in X : Bv = 0\}$$

the iteration remains in V. Therefore one can construct *conjugate directions* from the corrections in (3.7). In fact, defining the vector

$$g^{\ell} := Au^{\ell} + B^{\mathrm{T}}p^{\ell} - f$$

as the residual of the first block and computing h^{ℓ} from

$$\begin{pmatrix} \alpha I & B^{\mathrm{T}} \\ B & \end{pmatrix} \begin{pmatrix} h^{\ell} \\ p^{\ell} \end{pmatrix} = \begin{pmatrix} g^{\ell} \\ 0 \end{pmatrix},$$

we obtain the next conjugate direction and the next iterate from

$$\begin{aligned} d^\ell &:= -h^\ell + \beta_\ell d^{\ell-1} \\ u^{\ell+1} &:= u^\ell + \alpha_\ell d^\ell. \end{aligned}$$

The factors α_ℓ and β_ℓ are determined as in any cg-algorithm. Note that by construction $Bh^\ell = 0$ so that also

$$Bd^\ell = 0, \quad \ell = 0, 1, \ldots. \tag{3.9}$$

Thus one considers the cg-method confined to a subspace where A is definite. The cg-method based on (3.7) will be employed as a smoother in the cascadic multigrid algorithm in accordance with the concept for scalar equations in [1,8].

4. The cascadic multigrid iteration

Our objective is to analyse the following

CASCADIC Multigrid Algorithm:
Compute the exact solution u_0, q_0 of (2.3) on level $i = 0$. Set $v_0 := u_0$.
For $i = 1, \ldots, J : \{$

- Compute w_i as the prolongation of v_{i-1}.
- Compute $v^0 := v_i^0$ as the projection of w_i to $V_i := \ker B_i$.
- Execute $m = m_i$ steps of the cg-method.
- Set $v_i := v^m$

$\}$

Since the spaces X_i, M_i are nested, the prolongation

$$v_{i-1} \mapsto w_i, \quad q_{i-1} \mapsto q^0$$

in the above scheme is simply the inclusion. However, although each v_{i-1}^ℓ and hence v_{i-1} belong to V_{i-1}, its prolongation w_i will generally *not* belong to V_i. The correction can be performed by solving the sytem

$$\begin{pmatrix} C & B^{\mathrm{T}} \\ B & \end{pmatrix} \begin{pmatrix} \tilde{w} \\ \tilde{q} \end{pmatrix} = \begin{pmatrix} 0 \\ -Bw_i \end{pmatrix}, \tag{4.1}$$

where we again suppress the level index i in the matrices B, C. In fact, by the remarks at the beginning of Sect. 3, $\tilde{w}$ minimizes the quadratic functional (Cv, v) under the constraint $B\tilde{w} = -Bw_i$. One easily confirms that

$$\tilde{w} = -C^{-1}B^{\mathrm{T}}(BC^{-1}B^{\mathrm{T}})^{-1}Bw_i. \tag{4.2}$$

Hence,

(4.3) $v^0 := w_i + \tilde{w} = (I - C^{-1}B^{\mathrm{T}}(BC^{-1}B^{\mathrm{T}})^{-1}B)w_i =: P_C w_i.$

Thus, since for $(u,v)_C := (u, Cv) = (Cv, u)$

$$\begin{aligned}(P_C z - z, w)_C &= -(C^{-1}B^{\mathrm{T}}(BC^{-1}B^{\mathrm{T}})^{-1}Bz, Cw) \\ &= ((BC^{-1}B^{\mathrm{T}})^{-1}Bz, Bw) = 0, \qquad \text{for all } \ w \in V_i,\end{aligned}$$

the mapping P_C is just the orthogonal projection to V_i with respect to the inner product $(\cdot,\cdot)_C$.

The most convenient choice for C is αI. Noting that $P_{\alpha I} = P_I =: P$ (see (3.6)) for any $\alpha > 0$ this gives rise to the orthogonal projector with respect to the standard L_2-inner product, i.e.,

(4.4) $$\|P\|_0 = 1.$$

For completeness, we note that there is also a bound with respect to the $\|\cdot\|_1$-norm.

Lemma 1. *For $P = P_i$ defined by (3.6) one has $\|P_i\|_1 \le c$ uniformly in $i \in \mathbb{N}$.*

Proof. It suffices to prove that $S = S_i := B^{\mathrm{T}}(BB^{\mathrm{T}})^{-1}B$ is uniformly bounded in $\|\cdot\|_1$. To this end, note that $V = \ker B$ is a closed subspace of $H_0^1(\Omega)^d$. Therefore its orthogonal complement $V^\perp$ with respect to $(\cdot,\cdot)_1$ exists. Thus any $v \in X_i$ can be written as $v = z + w$ with $Bz = 0$ and $w \in V^\perp$. Obviously, $BSv = Bv = Bw$. Hence,

(4.5) $$\|Sv\|_1^2 = \|S(z+w)\|_1^2 = \|Sw\|_1^2.$$

On the other hand, since

(4.6) $$\|w\|_1^2 \sim (Bw, Bw) \quad \text{for } \ w \in V^\perp,$$

cf. Remark III.5.5 in [3], we obtain

$$\|Sw\|_1^2 \sim (BSw, BSw) = (Bw, Bw) \sim \|w\|_1^2 \le \|w\|_1^2 + \|z\|_1^2 = \|v\|_1^2,$$

and the assertion follows from (4.5). □

5. The cg-method and optimal polynomials

According to (3.8) in [4] the error in the v-component for the iteration (3.7) is given by

$$u - v^{\ell+1} = P(I - \frac{1}{\alpha_\ell}A)(u - v^\ell) = (I - \frac{1}{\alpha_\ell}PAP)(u - v^\ell), \tag{5.1}$$

where P is defined by (3.6). From the theory of the cg-method we know that

$$\begin{aligned} \|u - v^m\|_1 &\le c\,|||\,u - v^m|||\\ &= c\,\inf\left\{|||\,u - Q_m(PAP)v^0|||\ : \deg Q_m \le m,\ Q_m(0) = 1\right\}. \end{aligned} \tag{5.2}$$

Here the energy norm $|||\ \cdot\ |||$ is defined by $|||\,v|||\ := (v, PAPv)_0^{1/2}$, so that in the case of the Stokes problem

$$|||\,v|||\ := |Pv|_1. \tag{5.3}$$

It has been shown by Shaidurov [8] that, given $m \in \mathbb{N}$ and $\Lambda > 0$, there exists a polynomial Q_m such that

$$\begin{aligned} Q_m(0) &= 1\\ |\sqrt{x}Q_m(x)| &\le \tfrac{\sqrt{\Lambda}}{2m+1} \quad \text{for } x \in [0, \Lambda],\\ |Q_m(x)| &\le 1 \qquad\quad \text{for } x \in [0, \Lambda]. \end{aligned} \tag{5.4}$$

We set $\lambda_{\max} := \lambda_{\max}(PAP)$. Since PAP is selfadjoint, following Shaidurov [8] we obtain from (5.4) an operator Q_m with

$$\begin{aligned} |||\,Q_m v|||\ &\le \frac{\sqrt{\lambda_{\max}}}{2m+1}\|v\|_0,\ v \in X_i,\\ |||\,Q_m v|||\ &\le |||\,v|||\,, \qquad\quad v \in X_i. \end{aligned}$$

We emphasize that the energy norm is a mesh-dependent norm, since the projector in (5.3) depends on the grid. Therefore we reformulate the above bounds. Since $Q_m v$ belongs to V_i, we have $|||\,Q_m v|||\ = |Q_m v|_1$, i.e.

$$|Q_m v|_1 \le \frac{\sqrt{\lambda_{\max}}}{2m+1}\|v\|_0,\ v \in X_i, \tag{5.5}$$

$$|Q_m v|_1 \le |Pv|_1, \qquad v \in X_i. \tag{5.6}$$

Although during the computations Q_m is only applied to functions in the kernel V_i, it is crucial for the analysis that the estimates hold for all $v \in X_i$. Moreover we note that the inverse inequality (2.5) implies

$$\lambda_{\max} = \lambda_{\max}(PAP) \le \lambda_{\max}(A) \le ch^{-2}. \tag{5.7}$$

6. A recursion relation and final estimates

In contrast to cascadic iterations for scalar elliptic problems the prolongation of the approximate solution v_{i-1} on level $i-1$ is followed by a correction which projects the prolongated w_i to $v^0 := P_i w_i \in V_i$. Since P_i is an orthogonal projector relative to the L_2-inner product and therefore generally does not have norm one in H^1, the relation

$$\|v^0 - u_i\|_1 = \|P_i(w_i - u_i)\|_1$$

does not allows us to directly infer the estimate

$$\|v^0 - u_i\|_1 \le \|w_i - u_i\|_1$$

which would be needed in a convergence analysis following the concepts of [1,8]. The subsequent discussion indicates the corresponding difficulties. We will overcome them by switching to the $|\cdot|_1$-projector in the analysis. The terms which arise from the compensation will be estimated by applying the following lemma.

Lemma 2. *There exists a linear mapping $R_i : X_i \to V_i$ and a constant c such that*

$$\|(I - R_i)z_i\|_0 \le ch_i\|z_i\|_1 \quad \text{for all } z_i \in V_{i-1}, \tag{6.1}$$

$$|R_i z_i|_1 \le |z_i|_1 \qquad \text{for all } z_i \in X_i. \tag{6.2}$$

The proof of the lemma will be given in the next section.
First, under the regularity assumption

$$\|u\|_2 + \|p\|_1 \le c\|f\|_0,$$

one obtains the L_2-estimate

$$\|u - u_i\|_0 \le ch_i^2\|f\|_0.$$

Thus the triangle inequality yields

$$\|u_i - u_{i-1}\|_0 \le \|u_i - u\|_0 + \|u - u_{i-1}\|_0 \le ch_i^2\|f\|_0. \tag{6.3}$$

Here we assume as usual that h_i/h_{i-1} remains bounded.

We are now prepared to analyze the error produced by the scheme described in Sect. 4. To this end, we recall that u_i denotes the exact solution of the discrete problem in X_i, while $v^0 = v_i^0 := P_i w_i$ denotes the starting value for the iteration on the level i. As before w_i is the prolongation of the approximate solution $v_{i-1} = v^{m_{i-1}}$ in X_{i-1}. Abbreviating $Q_i := Q_{m_i}(P_i A_i P_i)$, we obtain

$$\begin{aligned} u_i - v_i &= Q_i(u_i - P_i v_{i-1}) \\ &= Q_i(u_i - P_i u_{i-1}) + Q_i P_i(u_{i-1} - v_{i-1}) \\ &= Q_i(u_i - P_i u_{i-1}) + Q_i R_i(u_{i-1} - v_{i-1}) \\ &\quad + Q_i(P_i - R_i)(u_{i-1} - v_{i-1}). \end{aligned} \tag{6.4}$$

As for the first summand, we invoke (5.5), (5.7), and (6.3) to obtain

$$\begin{aligned} |Q_i(u_i - P_i u_{i-1})|_1 &\le c\,\frac{h_i^{-1}}{m_i}\|u_i - P_i u_{i-1}\|_0 \\ &\le c\,\frac{h_i^{-1}}{m_i}\|u_i - u_{i-1}\|_0 \\ &\le c\,\frac{h_i}{m_i}\|f\|_0. \end{aligned} \tag{6.5}$$

The second term of (6.4) is estimated by employing (5.6), the estimate (6.2) in Lemma 2, and $P_i R_i = R_i$:

$$\begin{aligned} |Q_i R_i(u_{i-1} - v_{i-1})|_1 &\le |P_i R_i(u_{i-1} - v_{i-1})|_1 = |R_i(u_{i-1} - v_{i-1})|_1 \\ &\le |u_{i-1} - v_{i-1}|_1. \end{aligned} \tag{6.6}$$

The third summand on the right hand side of (6.4) represents the essential distinction from the scalar elliptic case. It reflects the nonconformity of the prolongation. Applying first (5.5), using as before that $P_i - R_i = P_i(I - R_i)$ and bearing (4.4) in mind, provides

$$\begin{aligned} |Q_i(P_i - R_i)(u_{i-1} - v_{i-1})|_1 &\le c\frac{h_i^{-1}}{m_i}\|(P_i - R_i)(u_{i-1} - v_{i-1})\|_0 \\ &\le c\frac{h_i^{-1}}{m_i}\|(I - R_i)(u_{i-1} - v_{i-1})\|_0. \end{aligned}$$

At this point the main estimate (6.1) from Lemma 2 comes into play which yields

$$\|(I - R_i)(u_{i-1} - v_{i-1})\|_0 \le c h_i |u_{i-1} - v_{i-1}|_1.$$

This in turn implies by the previous estimate that

$$|Q_i(P_i - R_i)(u_{i-1} - v_{i-1})|_1 \le \frac{c}{m_i}|u_{i-1} - v_{i-1}|_1. \tag{6.7}$$

By combining all three estimates (6.5), (6.6) and (6.7) for the terms on the right hand side of (6.4), we obtain immediately the recursion relation in the following proposition.

Proposition 3. *There exists a constant c such that*

(6.8) $$|u_i - v_i|_1 \le c\frac{h_i}{m_i}\|f\|_0 + (1 + \frac{c}{m_i})\,|u_{i-1} - v_{i-1}|_1.$$

The choice of m_i for the number of smoothing steps on level i was made in [1] such that the term for $i = J$ dominates in the sums $\sum_i m_i^{-1} h_i$ and $\sum_i m_i h_i^{-d}$. (Specifically the choice in [1] corresponds to setting $\alpha := (d+1)/2$ in the next theorem.) Therefore the error of the solution u_J and the computing effort are given by the contributions of the finest grids.

With the aid of Proposition 3 we are now in a position to establish similar properties here and show that the cascadic multigrid algorithm for the Stokes problem behaves like the cascadic algorithms investigated by Bornemann and Deuflhard [1] and by Shaidurov [8].

Theorem 4. *Assume that $1 < \alpha < d$ and that the CASCADIC multigrid algorithm described in Sect. 4 is applied with m_i cg steps on the levels $1 \le i \le J$ the m_i being the smallest integers satisfying*

(6.9) $$m_i \ge m_J\, 2^{\alpha(J-i)}.$$

Then the algorithm yields an approximate solution v_J on the highest level with

(6.10) $$\|u_J - v_J\|_1 \le c\frac{h_J}{m_J}\|f_0\|_0,$$

where the constant c is independent of f and J. Moreover, the complexity of the algorithm is bounded by $c m_J \dim X_J$.

Proof. Since $v_0 = u_0$, by Proposition 3 we obtain

$$\|u_J - v_J\|_1 \le c\sum_{j=0}^{J}\prod_{i=0}^{j-1}\left(1 + \frac{c}{m_{J-i}}\right)\frac{h_{J-j}}{m_{J-j}}\|f\|_0.$$

From (6.9) we infer that

$$\sum_{i=0}^{J-1}\frac{1}{m_{J-i}} \le \frac{1}{m_J}\sum_{i=0}^{J-1} 2^{-\alpha i} \le \frac{2}{m_J}.$$

Thus the products $\prod_{i=0}^{J-1}\left(1 + \frac{c}{m_{J-i}}\right)$ are uniformly bounded by $\exp(2c/m_J)$ and

(6.11) $$\|u_J - v_J\|_1 \le c\sum_{j=0}^{J}\frac{h_j}{m_j}\,\|f\|_0.$$

The estimate (6.10) now follows from (6.11) combined with Lemma 1.3 in [1], while for the above choice of the m_i the complexity estimate is a consequence of Lemma 1.4 in [1]. □

7. Proof of Lemma 2

Given $z_i \in X_i$, let $w_i \in X_i$ be the solution of

$$\begin{aligned} a(w_i, v) + b(v, p_i) &= a(z_i, v) \text{ for all } v \in X_i, \\ b(w_i, q) &= 0 \qquad\quad \text{for all } q \in M_i. \end{aligned} \tag{7.1}$$

Obviously we obtain a linear projection $R_i : X_i \to V_i$ if we set $R_i z_i := w_i$.

Since the finite element spaces X_i, M_i are stable, we have

$$\|w_i\|_1 + \|p_i\|_0 \le c\|z_i\|_1. \tag{7.2}$$

In order to apply Nitsche's trick, we consider the auxiliary variational problem

$$\begin{aligned} a(y, v) + b(v, r) &= (w_i - z_i, v)_0 \text{ for all } v \in H_0^1(\Omega)^d, \\ b(y, q) &= 0 \qquad\qquad\quad \text{for all } q \in L_{2,0}(\Omega). \end{aligned} \tag{7.3}$$

Since we have assumed H^2-regularity, we obtain

$$\|y\|_2 + \|r\|_1 \le c\|w_i - z_i\|_0. \tag{7.4}$$

Now we insert $v := w_i - z_i$ and $q := p_i$ into (7.3):

$$(w_i - z_i, w_i - z_i)_0 = a(y, w_i - z_i) + b(w_i - z_i, r) + b(y, p_i). \tag{7.5}$$

Note that the left hand side equals $\|w_i - z_i\|_0^2$. Next we recall that (7.1) holds for $v \in X_{i-1}$ and $q \in M_{i-1}$. Since we are only interested in estimates for $z_i \in V_{i-1}$, it follows that $b(z_i, q) = 0$ for $q \in M_{i-1}$. Hence,

$$\begin{aligned} a(w_i - z_i, v) + b(v, p_i) &= 0 \text{ for all } v \in X_{i-1}, \\ b(w_i - z_i, q) &= 0 \text{ for all } q \in M_{i-1}. \end{aligned} \tag{7.6}$$

The general approximation results for affine families of finite elements [3] guarantee that there is a $y_{i-1} \in X_{i-1}$ such that $\|y - y_{i-1}\|_1 \le ch\|y\|_2$ and $r_{i-1} \in M_{i-1}$ such that $\|r - r_{i-1}\|_0 \le ch\|r\|_1$. Combining this with (7.5) and (7.6), we obtain

$$\begin{aligned} &\|w_i - z_i\|_0^2 \\ &\quad = a(y - y_{i-1}, w_i - z_i) + b(w_i - z_i, r - r_{i-1}) + b(y - y_{i-1}, p_i) \\ &\quad \le c\|w_i - z_i\|_1 \left(\|y - y_{i-1}\|_1 + \|r - r_{i-1}\|_0\right) + c\|y - y_{i-1}\|_1\|p_i\|_0 \\ &\quad \le ch\|w_i - z_i\|_1 \left(\|y\|_2 + \|r\|_1\right) + ch\|y\|_2\|p_i\|_0. \end{aligned}$$

The triangle inequality and (7.2) yields $\|w_i - z_i\|_1 \le c\|z_i\|_1$. Finally, we use (7.2) for estimating $\|p_i\|_0$ and (7.4) for estimating $\|y\|_2$ and $\|r\|_1$:

$$\|w_i - z_i\|_0^2 \le ch\|z_i\|_1 \, \|w_i - z_i\|_0. \tag{7.7}$$

We divide (7.7) by $\|w_i - z_i\|_0$, set $R_i z_i := w_i$, and the proof of (6.1) is complete.

In order to prove (6.2) we set $v := w_i$, $q := p_i$ in (7.1) and obtain

$$|w_i|_1^2 = a(w_i, w_i) = a(z_i, w_i) + 0 \le |z_i|_1 \, |w_i|_1.$$

After dividing by $|w_i|_1$ we have (6.2). □

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