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A Multigrid Method for Nonconforming FE-Discretisations with Application to Non-Matching Grids

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Abstract

Nonconforming finite element discretisations require special care in the construction of the prolongation and restriction in the multigrid process. In this paper, a general scheme is proposed, which guarantees the approximation property. As an example, the technique is applied to the discretisation by non-matching grids (mortar elements).

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1. Introduction

Recently, domain decomposition methods have been applied to situations where subdomain meshes may be separately constructed and are non-matching along the interfaces. The method was called *mortar element method* in [3]. When this scheme is employed with finite elements, it may be considered as a nonconforming method or as a mixed method.

In this paper, we will treat the mortar elements in the framework of nonconforming methods, and we assume that the Lagrange multipliers have been eliminated as in the setting of the second author [9]. When multigrid methods are designed, there is now the problem that the finite element spaces are not nested.

Therefore, we have to construct appropriate prolongation operators. In the multigrid scheme for other elements, as, e.g., for the Crouzeix–Raviart elements in [5, 8] the L_2-projectors could be chosen for the prolongations. We will abandon this restriction and describe a more general framework which admits a lot of freedom in the construction. In particular, a prolongation that is natural for the mortar elements fits into our framework. The approximation property for the convergence proof will be derived from an auxiliary problem. In essence, we will only assume that an L_2 error estimate is known for the finite elements under consideration.

In Section 2 we recall some notation for nonconforming finite elements. Section 3 is concerned with an extension of the prolongation operators which admits in Section 4 to derive the central approximation property from an L_2 error estimate. In Section 5 the associated smoothing property and the convergence is discussed. Section 6 provides the application to mortar elements in the geometric conforming case. We conclude with a generalisation to geometric nonconforming meshes.

After completing the paper, we learnt about the paper [11] investigating the mortar finite element method by other theoretical means.

2. Multigrid Transfer

2.1. Variational Problem

We consider a variational problem of the following form. Let $\mathcal{H}^1$ be a Hilbert space. Given a bilinear form $a(\cdot,\cdot)$ on $\mathcal{H}^1 \times \mathcal{H}^1$ and a functional $f \in \mathcal{H}^{-1} := (\mathcal{H}^1)'$, we look for a solution $u \in \mathcal{H}^1$ of

$$a(u, w) = f(w) \qquad \text{for all } w \in \mathcal{H}^1. \tag{2.1}$$

Let

$$\mathcal{H}^{-1} \subset \mathcal{H}^0 \subset \mathcal{H}^1$$

be the Gelfand triple, e.g., $\mathcal{H}^1 := H_0^1(\Omega)$, $\mathcal{H}^0 := L_2(\Omega)$, and $\mathcal{H}^{-1} := H^{-1}(\Omega)$. In addition, we need a space $\mathcal{H}^2 \subset \mathcal{H}^1$ (e.g., $\mathcal{H}^2 = H^2(\Omega) \cap H_0^1(\Omega)$). The norms of $\mathcal{H}^k$ are denoted by $\|\cdot\|_k$. The scalar product in $\mathcal{H}^0$ is written as $(\cdot,\cdot)_0$.

We assume:
(i) Solvability: For all $f \in \mathcal{H}^{-1}$, (2.1) has a unique solution $u \in \mathcal{H}^1$ with $\|u\|_1 \le C\,\|f\|_{-1}$.
(ii) Regularity: If $f \in \mathcal{H}^0$, (2.1) has a solution $u \in \mathcal{H}^2$ with $\|u\|_2 \le C\,\|f\|_0$.

2.2. Nonconforming Discretisation

Let $V_\ell \subset \mathcal{H}^0$ for $\ell = 0, 1, \dots$ be a sequence of (nonconforming) finite element spaces, i.e., we do not assume that the spaces are nested. Instead of the bilinear form $a(\cdot,\cdot)$ a mesh-dependent bilinear form $a_\ell(\cdot,\cdot)$ on $V_\ell \times V_\ell$ is used. For $f \in \mathcal{H}^0$, (2.1) is discretised by

$$u_\ell \in V_\ell \quad \text{with} \quad a_\ell(u_\ell, w_\ell) = f(w_\ell) \qquad \text{for all } w_\ell \in V_\ell. \tag{2.2}$$

We assume that also (2.2) is solvable and that the error estimate

$$\|u - u_\ell\|_0 \le C_e h_\ell^{2m}\,\|u\|_2 \tag{2.3}$$

holds, cf. Braess [4, p. 102], Hackbusch [13, (8.4.15b)]. Here, $2m$ is the order of the differential operator, i.e., $\mathcal{H}^1$ is a subspace of $H^m(\Omega)$. As usual, h_ℓ is the size of the finite element mesh of V_ℓ.

Together with the regularity assumption $\|u\|_2 \le C\,\|f\|_0$ from above, we obtain

$$\|u - u_\ell\|_0 \le C_0 h_\ell^{2m}\,\|f\|_0\,. \tag{2.4}$$

2.3. Matrix Representation

Let $\{b_{\ell,i} : i \in I_\ell\}$ be a basis of V_ℓ, where I_ℓ is the corresponding index set (e.g., the set of nodal points). The coefficient vector space $\mathbb{R}^{I_\ell}$ is denoted by $\mathcal{U}_\ell$. The vectors in $\mathcal{U}_\ell$ are $\underline{u}_\ell = (u_{\ell,i})_{i\in I_\ell}$, and $\mathcal{U}_\ell$ will be equipped with the usual Euclidean norm $\|\cdot\|_{\mathcal{U}_\ell}$ (scaled by a suitable factor to ensure (2.6) below), so that the adjoint mappings are given by the transposed matrices (maybe up to a fixed factor).

The isomorphism between $\mathcal{U}_\ell$ and V_ℓ is denoted by ϕ_ℓ:

$$\phi_\ell : \mathcal{U}_\ell \to V_\ell \;\text{ with }\; u_\ell = \phi_\ell\,\underline{u}_\ell := \sum\nolimits_{i\in I_\ell} u_{\ell,i} b_{\ell,i}\,. \tag{2.5}$$

The finite element matrix A_ℓ corresponding to $a_\ell(\cdot,\cdot)$ has the coefficients $a_{\ell,ij} = a_\ell(b_{\ell,j}, b_{\ell,i})$. The variational problem (2.2) is equivalent to

$$A_\ell\,\underline{u}_\ell = \underline{f}_\ell$$

with

$$\underline{f}_\ell = \phi_\ell^* f, \quad \text{i.e.,} \quad f_{\ell,i} = f(b_{\ell,i}) = (f, b_{\ell,i})_0.$$

As mentioned above, after a suitable scaling we require the equivalence of the Euclidean norm $\|\cdot\|_{\mathcal{U}_\ell}$ and the $\mathcal{H}^0$-norm:

$$\frac{1}{C_{\phi^{-1}}}\|\underline{v}_\ell\|_{\mathcal{U}_\ell} \le \|v_\ell\|_0 \le C_\phi\|\underline{v}_\ell\|_{\mathcal{U}_\ell} \quad \text{for all } v_\ell = \phi_\ell\,\underline{v}_\ell. \tag{2.6}$$

2.4. General Concept for the Multigrid Prolongation

Main ingredients of the multigrid algorithm are the prolongation

$$p : \mathcal{U}_{\ell-1} \to \mathcal{U}_\ell$$

and the restriction $r = p^* : \mathcal{U}_\ell \to \mathcal{U}_{\ell-1}$.

In the case of a *conforming* finite element discretisation with a finite element hierarchy $V_0 \subset \ldots \subset V_{\ell-1} \subset V_\ell$, one obtains the following commutative diagram:

$$\begin{array}{ccc} V_{\ell-1} & \overset{inclusion}{\underset{\iota}{\hookrightarrow}} & V_\ell \\ \phi_{\ell-1} \uparrow & & \uparrow \phi_\ell \\ \mathcal{U}_{\ell-1} & \underset{p}{\longrightarrow} & \mathcal{U}_\ell \end{array}.$$

In this case, the *canonical* prolongation is given by $p = \phi_\ell^{-1} \circ \phi_{\ell-1}$.

In the following we will admit $V_{\ell-1} \not\subset V_\ell$, and the inclusion is to be replaced by a suitable mapping

$$\iota : V_{\ell-1} \to V_\ell. \tag{2.7}$$

Once ι has been given, we are able to define the prolongation and restriction by

$$p = \phi_\ell^{-1} \circ \iota \circ \phi_{\ell-1} \quad \text{and} \quad r = p^* = \phi_{\ell-1}^* \circ \iota^* \circ (\phi_\ell^*)^{-1}. \tag{2.8}$$

In the next section, we will propose a general construction of ι leading to the *approximation property*

$$\|A_\ell^{-1} - p\, A_{\ell-1}^{-1} r\|_{\mathcal{U}_\ell \leftarrow \mathcal{U}_\ell} \le C_A h_\ell^{2m}, \tag{2.9}$$

which is an essential sufficient condition for the multigrid convergence (cf. Hackbusch [12, §6.1.3]).

3. Construction of the Prolongation

3.1. Spaces Σ and S

Although the algorithm needs only the mapping $\iota : V_{\ell-1} \to V_\ell$ (cf. (2.7)), the theoretical consideration will lead to a variational problem (4.7) on the sum $V_{\ell-1} + V_\ell$ and require ι to be defined and bounded on $V_{\ell-1} + V_\ell$ (or on a larger space). Since $\iota : V_\ell \to V_\ell$ has to be the identity, we must construct a bounded mapping $\iota : V_{\ell-1} \to V_\ell$ such that its restriction to $V_{\ell-1} \cap V_\ell$ is the identity.

In order to make the metric structure of the sum more transparent we will refer to a (possibly larger) space Σ with

$$V_{\ell-1} + V_\ell \subset \Sigma \subset \mathcal{H}^0.$$

The space Σ and the space S defined below belong to the index pair $(\ell-1, \ell)$, and $\Sigma_{\ell-1,\ell}$ and $S_{\ell-1,\ell}$ would be a more precise notation. For the sake of simplicity, we omit these indices.

Here we also note that the sum $V_{\ell-1} + V_\ell$ plays an important role when Stevenson [14] considers an axiomatic framework for the Cascadic multigrid algorithms suitable for nonconforming elements.

Next, we need an auxiliary space S, which is connected with Σ and V_ℓ via the mappings σ and π, as shown in the following commutative diagram:

$$\begin{array}{ccc}
\Sigma & \xrightarrow{\sigma} & S \\
\text{inclusion}\uparrow & \nwarrow & \downarrow\pi \\
V_{\ell-1} & \xrightarrow[\iota]{} & V_\ell \\
\phi_{\ell-1}\uparrow & & \uparrow\phi_\ell \\
\mathcal{U}_{\ell-1} & \xrightarrow[p]{} & \mathcal{U}_\ell
\end{array}$$

.

The desired mapping ι (more precisely, its extension to Σ) is the product

$$\iota = \pi \circ \sigma : \ \Sigma \to V_\ell. \tag{3.1}$$

Before we will discuss the characteristic requirements concerning π, σ, and ι in the next subsection, for elucidating the formalism, we specify the spaces and mappings for the *Crouzeix-Raviart element*, i.e., for the simplest nonconforming finite elements (cf. Braess-Verfürth [5]).

Example 1. Let $\mathcal{T}_{\ell-1}$ be the coarse triangulation of the domain Ω, while $\mathcal{T}_\ell$ is obtained by regular halving of all triangle sides. V_ℓ is the space of all piecewise linear functions which are continuous at the midpoints of edges in $\mathcal{T}_\ell$. Define the nodal point set $\mathcal{N}_\ell$ by all midpoints of edges in $\mathcal{T}_\ell$ (except boundary points in the case of Dirichlet conditions). For all $\alpha \in \mathcal{N}_\ell$, basis functions $b_{\ell,\alpha} \in V_\ell$ are defined by $b_{\ell,\alpha}(\beta) = \delta_{\alpha\beta}$ $(\alpha, \beta \in \mathcal{N}_\ell)$ with the Kronecker symbol δ. Then, $\mathcal{U}_\ell := \ell_2(\mathcal{N}_\ell)$ is the coefficient space which is mapped by $\phi_\ell : (c_{\ell,\alpha})_{\alpha\in\mathcal{N}_\ell} \mapsto u_\ell = \sum_{\alpha\in\mathcal{N}_\ell} c_{\ell,\alpha} b_{\ell,\alpha}$ onto V_ℓ. Similarly, $V_{\ell-1}, \mathcal{U}_{\ell-1}$, and the isomorphism $\phi_{\ell-1}$ are defined.

An appropriate space Σ is the space of piecewise linear elements with respect to the fine triangulation $\mathcal{T}_\ell$ that may be discontinuous at the edges of this triangulation. Obviously, $V_{\ell-1} + V_\ell \subset \Sigma$.

We set $S := \mathcal{U}_\ell, \pi := \phi_\ell$, and define σ as follows: Every nodal point $\alpha \in \mathcal{N}_\ell$ is the midpoint of the common side of adjacent triangles T and T' from $\mathcal{T}_\ell$. For $\alpha \in \mathcal{N}_\ell$ we set

$$(\sigma v)_\alpha := \frac{1}{2}\left[v|_T(\alpha) + v|_{T'}(\alpha)\right].$$

Here, the linear function $v|_T$ is understood to be extended to $\bar{T}$.

3.2. Conditions on π and σ

The space Σ will be equipped with the norm and the scalar product of $\mathcal{H}^0$, and S is assumed to become a Hilbert space by a norm $\|\cdot\|_S$ and a scalar product $(\cdot,\cdot)_S$ whose specification may depend on the specific finite element space. The mapping $\sigma : \Sigma \to S$ is assumed to be *bounded*:

$$\|\sigma\|_{S \leftarrow \mathcal{H}^0} \le C_\sigma. \tag{3.2}$$

Furthermore, $\pi : S \to V_\ell$ is required to be *injective* and *bounded*:

$$\pi \text{ is } \textit{injective} \text{ and } \|\pi\|_{V_\ell \leftarrow S} \le C_\pi. \tag{3.3}$$

The product $\iota = \pi \circ \sigma$ from (3.1) is assumed to be a *projection* onto V_ℓ, i.e.,

$$\iota|_{V_\ell} = \pi \circ \sigma|_{V_\ell} = id : \; V_\ell \to V_\ell. \tag{3.4}$$

Remark 3.2. If the conditions (3.2)–(3.4) hold, then $\iota : \Sigma \to V_\ell \subset \Sigma$ is a bounded projection onto V_ℓ:

$$\|\iota\|_{\mathcal{H}^0 \leftarrow \mathcal{H}^0} \le C_\iota := C_\pi C_\sigma. \tag{3.5}$$

Moreover $\pi : S \to V_\ell$ is an isomorphism.

Proof: The boundedness (3.5) is a direct consequence of (3.2)–(3.3).

Since the range of π is V_ℓ, π is injective and surjective. Hence, $\pi^{-1} = \sigma|_{V_\ell}$, and π is an isomorphism. □

The introduction of the intermediate space S gives us more freedom in the construction of the mappings. Of course, in many cases the set S will coincide with V_ℓ or $\mathcal{U}_\ell$. We emphasise that only boundedness in $\mathcal{H}^0$ is required for ι, while the concept of Brenner [8, 15] also refers to conditions with respect to the energy norm.

4. Coarse-Grid Correction and Approximation Property

4.1. Coarse-Grid Correction

Let an approximation $\tilde{u}_\ell$ be given. Its defect $d_\ell \in (V_\ell)' = V_\ell$ is defined by

$$(d_\ell, w_\ell)_0 = a_\ell(\tilde{u}_\ell, w_\ell) - f(w_\ell) \qquad \text{for all } w_\ell \in V_\ell.$$

Using (2.2) and the error

$$e_\ell := \tilde{u}_\ell - u_\ell, \tag{4.1}$$

one obtains a characterisation of d_ℓ by

$$(d_\ell, w_\ell)_0 = a_\ell(e_\ell, w_\ell) \qquad \text{for all } w_\ell \in V_\ell. \tag{4.2}$$

The residue of the linear system is $\underline{d}_\ell = A_\ell \, \underline{e}_\ell$ with $e_\ell = \phi_\ell \, \underline{e}_\ell$. Because of $(A_\ell \, \underline{e}_\ell, \underline{w}_\ell)_{\mathcal{U}_\ell} = a_\ell(e_\ell, w_\ell)$ with $w_\ell = \phi_\ell \, \underline{w}_\ell$ and $(d_\ell, w_\ell)_0 = (d_\ell, \phi_\ell \, \underline{w}_\ell)_0 = (\phi_\ell^* d_\ell, \underline{w}_\ell)_{\mathcal{U}_\ell}$, we conclude

Remark 4.1. The residue

$$\underline{d}_\ell = A_\ell \, \underline{e}_\ell \in \mathcal{U}_\ell \tag{4.3}$$

has the representation $\underline{d}_\ell = \phi_\ell^* d_\ell$ with d_ℓ being defined in (4.2).

The coarse-grid correction $e_{\ell-1} \in V_{\ell-1}$ approximates the finite element function $e_\ell \in V_\ell$. It is determined as the solution of the coarse-grid equation

$$a_{\ell-1}(e_{\ell-1}, w_{\ell-1}) = a_\ell(e_\ell, \iota w_{\ell-1}) \qquad \text{for all } w_{\ell-1} \in V_{\ell-1}. \tag{4.4}$$

Here ι is the mapping specified in the previous section. Note that it is required for converting the function $w_{\ell-1}$ from $V_{\ell-1}$ into an element of V_ℓ. The correction yields the new approximation $u_\ell^{new} := \tilde{u}_\ell - \iota e_{\ell-1}$, cf. [5, 8]. The error after the coarse-grid correction is obviously

$$e_\ell^{new} = u_\ell^{new} - u_\ell = e_\ell - \iota e_{\ell-1}. \tag{4.5}$$

4.2. An Auxiliary Problem

We will estimate $\|e_\ell^{new}\|_0$ by constructing an auxiliary problem for which $e_{\ell-1}$ and e_ℓ are the finite element solutions at the levels $\ell - 1$ and ℓ, respectively. To this end we introduce two Riesz representations of the residue.

Given e_ℓ, define $\bar{r}_\ell \in S$ by

$$(\bar{r}_\ell, \bar{w})_S = a_\ell(e_\ell, \pi \bar{w}) \qquad \text{for all } \bar{w} \in S. \tag{4.6}$$

Similarly, let $g \in \Sigma$ be the solution of

$$(g, s)_0 = (\bar{r}_\ell, \sigma s)_S \qquad \text{for all } s \in \Sigma. \tag{4.7}$$

Lemma 4.2. *g has the representation $g = \sigma^* \bar{r}_\ell = \iota^* (\phi_\ell^*)^{-1} \underline{d}_\ell$ and satisfies*

$$\|g\|_0 \le C_\sigma \|\bar{r}_\ell\|_S \le C_\sigma C_{\phi^{-1}} C_\pi \|\underline{d}_\ell\|_{\mathcal{U}_\ell}. \tag{4.8}$$

Proof: First, we have $\bar{r}_\ell = \pi^* (\phi_\ell^*)^{-1} \underline{d}_\ell$ since $(\pi^* (\phi_\ell^*)^{-1} \underline{d}_\ell, \bar{w})_S = (\pi^* d_\ell, \bar{w})_S = (d_\ell, \pi \bar{w})_0 = a_\ell(e_\ell, \pi \bar{w})$ for all $\bar{w} \in S$. Combining this with $g = \sigma^* \bar{r}_\ell$ and $\iota^* = \sigma^* \pi^*$ we obtain the required representation of g.

Moreover, the inequality

$$\begin{aligned}
\|\bar{r}_\ell\|_S^2 = (\bar{r}_\ell, \bar{r}_\ell)_S \underset{(4.6)}{=} a_\ell(e_\ell, \pi \bar{r}_\ell) \underset{(4.2)}{=} \\
= (d_\ell, \pi \bar{r}_\ell)_0 \underset{\text{Remark 4.1}}{=} ((\phi_\ell^*)^{-1} \underline{d}_\ell, \pi \bar{r}_\ell)_0 = (\underline{d}_\ell, \phi_\ell^{-1} \pi \bar{r}_\ell)_{\mathcal{U}_\ell} \\
\le \|\underline{d}_\ell\|_{\mathcal{U}_\ell} \|\phi_\ell^{-1} \pi \, \bar{r}_\ell\|_{\mathcal{U}_\ell} \le \|\underline{d}_\ell\|_{\mathcal{U}_\ell} C_{\phi^{-1}} C_\pi \, \|\bar{r}_\ell\|_S
\end{aligned}$$

yields $\|\bar{r}_\ell\|_S \le C_{\phi^{-1}} C_\pi \|\underline{d}_\ell\|_{\mathcal{U}_\ell}$. Next we estimate $\|g\|_0^2 = (g, g)_0 = (\bar{r}_\ell, \sigma g)_S \le \|\bar{r}_\ell\|_S \, \|\sigma g\|_S \le \|\bar{r}_\ell\|_S \, C_\sigma \, \|g\|_0$. After dividing by $\|g\|_0$ and inserting the estimate of $\bar{r}_\ell$ above, we obtain the required inequality. □

Although the mapping ι need only be defined on $V_{\ell-1}$ for the computations, we have extended it to $V_{\ell-1} + V_\ell$. The aim of that process is an interesting property of g which is the subject of

Proposition 4.3. *The variational problem in $\mathcal{H}^1$,*

$$a(z, w) = (g, w)_0 \qquad \textit{for all } w \in \mathcal{H}^1,$$

has the finite element solutions $e_{\ell-1}$ and e_ℓ from (4.4) and (4.1) at the levels $\ell - 1$ and ℓ, respectively.

Proof: a) On level ℓ, we conclude from (3.4), (4.6), and (4.7) that

$$a_\ell(e_\ell, w_\ell) = a_\ell(e_\ell, \pi \sigma w_\ell) = (\bar{r}_\ell, \sigma w_\ell)_S = (g, w_\ell)_0 \qquad \text{for all } w_\ell \in V_\ell.$$

b) On level $\ell - 1$, it follows from (4.4) and (3.1) that we have for all $w_{\ell-1} \in V_{\ell-1}$

$$a_{\ell-1}(e_{\ell-1}, w_{\ell-1}) = a_\ell(e_\ell, \iota w_{\ell-1}) = a_\ell(e_\ell, \pi \sigma w_{\ell-1}) = (g, w_{\ell-1})_0.$$

The last equality was obtained as in part a) by (4.6) and (4.7). □

4.3. Approximation Property

Let $z \in \mathcal{H}^1$ be the solution of the variational problem in Proposition 4.3. Because of $g \in \mathcal{H}^0$ and the regularity condition in Section 2.1, z belongs to $\mathcal{H}^2$. The error estimate (2.4) yields the statement that z and its finite element approximations $e_{\ell-1}$ and e_ℓ satisfy

$$\|z - e_j\|_0 \le C_0 h_j^{2m} \|g\|_0 \quad \text{for } j = \ell - 1, \ell. \tag{4.9}$$

The error $e_\ell^{new} = \phi_\ell\, \underline{e}_\ell^{new}$ from (4.5) after the coarse-grid correction will be estimated in the following proposition. Here, we make use of the standard assumption (on the mesh size ratio)

$$h_{\ell-1} \le C_h h_\ell,$$

which usually holds with $C_h = 2$.

Proposition 4.4. *Under the previous assumptions, the estimate*

$$\|\underline{e}_\ell^{new}\|_{\mathcal{U}_\ell} \le C_A h_\ell^{2m} \|\underline{d}_\ell\|_{\mathcal{U}_\ell} \tag{4.10}$$

holds for all $\underline{d}_\ell \in \mathcal{U}_\ell$ *with* $C_A := C_{\phi^{-1}}^2 C_\iota C_0 (1 + C_h^{2m}) C_\sigma C_\pi$.

Proof: $\|\underline{e}_\ell^{new}\|_{\mathcal{U}_\ell} \le C_{\phi^{-1}} \|e_\ell^{new}\|_0$ holds because of (2.6). From (3.4) and (3.5) it follows that

$$\|e_\ell^{new}\|_0 = \|e_\ell - \iota e_{\ell-1}\|_0 = \|\iota(e_\ell - e_{\ell-1})\|_0 \le C_\iota \|e_\ell - e_{\ell-1}\|_0 . \tag{4.11}$$

Now (4.9) implies

$$\|e_\ell - e_{\ell-1}\|_0 \le \|e_\ell - z\|_0 + \|z - e_{\ell-1}\|_0 \le C_0 (h_\ell^{2m} + h_{\ell-1}^{2m}) \|g\|_0 .$$

Moreover, $h_\ell^{2m} + h_{\ell-1}^{2m} \le (1 + C_h^{2m}) h_\ell^{2m}$.

Finally, we insert (4.8) to obtain

$$\|e_\ell - e_{\ell-1}\|_0 \le C_{ee} h_\ell^{2m} \|\underline{d}_\ell\|_{\mathcal{U}_\ell} \tag{4.12}$$

with $C_{ee} := C_0 C_\sigma C_{\phi^{-1}} C_\pi (1 + C_h^{2m})$. After inserting this estimate into (4.11) the proof is complete. □

In order to derive the desired approximation property (2.9) from the error estimate, we return to the vector representation of the coarse-grid correction

$$\underline{e}_{\ell-1} = A_{\ell-1}^{-1} \phi_{\ell-1}^* g = A_{\ell-1}^{-1} \phi_{\ell-1}^* \sigma^* \pi^* (\phi_\ell^*)^{-1} \underline{d}_\ell = A_{\ell-1}^{-1} \phi_{\ell-1}^* \iota^* (\phi_\ell^*)^{-1}\, \underline{d}_\ell \tag{4.13}$$

(cf. Lemma 4.2). Therefore, the representation of $\underline{e}_\ell^{new}$ is

$$\begin{aligned}\underline{e}_\ell^{new} &= \phi_\ell^{-1} e_\ell^{new} = \phi_\ell^{-1}(e_\ell - \iota e_{\ell-1}) = \underline{e}_\ell - \phi_\ell^{-1}\,\iota\phi_{\ell-1}\underline{e}_{\ell-1} \\ &= A_\ell^{-1}\underline{d}_\ell - \phi_\ell^{-1}\,\iota\,\phi_{\ell-1}A_{\ell-1}^{-1}\phi_{\ell-1}^{*}\iota^{*}(\phi_\ell^{*})^{-1}\underline{d}_\ell \\ &= (A_\ell^{-1} - pA_{\ell-1}^{-1}r)\,\underline{d}_\ell\end{aligned}$$

with p and r from (2.8). The inequality (4.10) is equivalent to (2.9). This proves

Proposition 4.5. *Under the required assumptions, the approximation property (2.9) holds with the constant* $C_A := C_{\phi^{-1}}^2 C_\iota C_0(1 + C_h^{2m})C_\sigma C_\pi$.

Finally we note that the approximation property for the framework in [4, p. 222] is obtained from Proposition 4.4 and (4.1),

$$\|e_\ell - \iota e_{\ell-1}\|_0 \le C_A h_\ell^{2m}\|A_\ell \underline{e}_\ell\|_{\mathcal{U}_\ell}.$$

5. Smoothing Property and Multigrid Convergence

5.1. Smoothing Property

It is well known that the convergence of multigrid algorithms can only be proved if there is an *inverse property* which fits to the error estimates in Section 2.2. Specifically, we assume that the matrix A_ℓ is bounded by

$$\|A_\ell\|_{\mathcal{U}_\ell \leftarrow \mathcal{U}_\ell} \le C_B h_\ell^{-2m}. \tag{5.1}$$

If, in addition, A_ℓ is positive definite, the simplest possible iteration (the Richardson iteration) is already a smoothing iteration:

$$u_\ell \mapsto \mathcal{S}_\ell(u_\ell, f_\ell) := u_\ell - C_B^{-1}h_\ell^{2m}(A_\ell u_\ell - f_\ell),$$
$$\mathcal{S}_\ell := I - C_B^{-1}h_\ell^{2m}A_\ell,$$

since it satisfies the smoothing property

$$\|A_\ell \mathcal{S}_\ell^\nu\|_{\mathcal{U}_\ell \leftarrow \mathcal{U}_\ell} \le \eta(\nu)h_\ell^{-2m} \quad \text{for all } \nu \ge 0 \tag{5.2}$$

with $\eta(\nu) := C_B\eta_0(\nu)$ and $\eta_0(\nu) := \nu^\nu/(\nu+1)^{(\nu+1)}$ (cf. [12, §6.2]). The cases of A_ℓ not being positive definite or of other smoothing iterations are described in [12], too.

In addition, we assume that $\mathcal{S}_\ell^\nu$ remains bounded:

$$\|\mathcal{S}_\ell^\nu\|_{\mathcal{U}_\ell \leftarrow \mathcal{U}_\ell} \le C_S \qquad \text{for all } \nu \ge 0,\ \ell \ge 1. \tag{5.3}$$

Under the previous assumptions, the Richardson iteration satisfies (5.3) with $C_S = 1$.

5.2. Multigrid Convergence

The iteration matrix of the two-grid iteration (with pre-smoothing, only) equals

$$M_\ell^{TGM}(\nu) = (A_\ell^{-1} - p\,A_{\ell-1}^{-1}\,r)(A_\ell\,\mathcal{S}_\ell^\nu),$$

where ν is the number of smoothing iterations described by the iteration matrix $\mathcal{S}_\ell$. The approximation property (2.9) and the smoothing property (5.2) yield

$$\|M_\ell^{TGM}(\nu)\|_{\mathcal{U}_\ell \leftarrow \mathcal{U}_\ell} \le C_A\,\eta(\nu) \qquad \text{for all } \nu \ge 0.$$

Let $\zeta < 1$ be given. Since $\lim_{\nu\to\infty} \eta(\nu) = 0$, the right-hand side is bounded by $C_A\,\eta(\nu) \le \zeta < 1$ for all $\nu \ge \underline{\nu}(\zeta)$, where $\underline{\nu}(\zeta)$ is independent of the mesh size h_ℓ. In the following, the number ν is fixed.

Replacing the exact solution of the coarse-grid problem by two iterations of the multigrid method, we obtain the W-cycle. Its iteration matrix $M_\ell^{MGM} = M_\ell^{MGM}(\nu)$ is given by the recursion

$$\begin{aligned} M_1^{MGM} &= M_1^{TGM}, \\ M_\ell^{MGM} &= M_\ell^{TGM} + p(M_{\ell-1}^{MGM})^2 A_{\ell-1}^{-1} r A_\ell \mathcal{S}_\ell^\nu \quad \text{for } \ell \ge 2. \end{aligned} \tag{5.4}$$

Let $\zeta_\ell := \|M_\ell^{MGM}\|_{\mathcal{U}_\ell \leftarrow \mathcal{U}_\ell}$. From the two-grid analysis we know $\zeta_1 \le \zeta < 1$. Equation (5.4) together with the estimates $\|p\|_{\mathcal{U}_\ell \leftarrow \mathcal{U}_{\ell-1}} \le C_\phi C_\iota C_{\phi^{-1}} =: C_p$ (cf. (2.8)) yields the recursive inequality

$$\zeta_\ell \le \zeta + C_p \zeta_{\ell-1}^2 \|A_{\ell-1}^{-1}\,r\,A_\ell\,\mathcal{S}_\ell^\nu\|_{\mathcal{U}_{\ell-1} \leftarrow \mathcal{U}_\ell}. \tag{5.5}$$

The essential step is to prove

$$\|A_{\ell-1}^{-1}\,r\,A_\ell\,\mathcal{S}_\ell^\nu\|_{\mathcal{U}_{\ell-1} \leftarrow \mathcal{U}_\ell} \le C, \tag{5.6}$$

since then (5.5) implies $\zeta_\ell \le \zeta + C^* \zeta_{\ell-1}^2$. Note that we have $\zeta < 1/4C^*$ if the number of smoothing steps ν is sufficiently large. Now the recursion relation implies $\zeta_\ell \le 2\zeta/(1+\sqrt{1-4\zeta C^*})$ (cf. [12, (7.1.7e)]). This proves mesh independent convergence rates for sufficiently many smoothing steps ν.

In [12, Lemma 7.1.5], the estimate (5.6) is established under the condition $\|\underline{u}_{\ell-1}\|_{\mathcal{U}_{\ell-1}} \le \underline{C}_p \|p\underline{u}_{\ell-1}\|_{\mathcal{U}_\ell}$. Usually, this inequality is valid, but here it is not possible to prove this estimate via assumptions on σ, since, typically, σ is not injective on Σ. Therefore, we provide a different proof. Using $\|A_{\ell-1}^{-1}\, r\, A_\ell\, \mathcal{S}_\ell^\nu\|_{\mathcal{U}_{\ell-1}\leftarrow\mathcal{U}_\ell} \le C_S \|A_{\ell-1}^{-1}\, r\, A_\ell\|_{\mathcal{U}_{\ell-1}\leftarrow\mathcal{U}_\ell}$ (cf. (5.3)), it remains to estimate $A_{\ell-1}^{-1}\, r A_\ell$.

Equation (4.13) can be rewritten as $\underline{e}_{\ell-1} = A_{\ell-1}^{-1} r\, \underline{d}_\ell = A_{\ell-1}^{-1}\, r A_\ell\, \underline{e}_\ell$. Combining (4.12) with $\underline{d}_\ell = A_\ell\, \underline{e}_\ell$ and (5.1), we infer $\|e_\ell - e_{\ell-1}\|_0 \le C_{ee} C_B \|\underline{e}_\ell\|_{\mathcal{U}_\ell}$. This shows

$$\|e_{\ell-1}\|_0 \le \|e_{\ell-1} - e_\ell\|_0 + \|e_\ell\|_0 \le (C_{ee}C_B + C_\phi)\|\underline{e}_\ell\|_{\mathcal{U}_\ell}$$

and $\|\underline{e}_{\ell-1}\|_{\mathcal{U}_{\ell-1}} \le C_{\phi^{-1}}(C_{ee}C_B + C_\phi)\|\underline{e}_\ell\|_{\mathcal{U}_\ell}$. Since this inequality holds for all $\underline{e}_\ell \in \mathcal{U}_\ell$, the representation $\underline{e}_{\ell-1} = A_{\ell-1}^{-1} r\, A_\ell\, \underline{e}_\ell$ yields the next lemma.

Lemma 5.1. *Under the previous conditions,* $\|A_{\ell-1}^{-1} r A_\ell\|_{\mathcal{U}_{\ell-1}\leftarrow\mathcal{U}_\ell} \le C$ *holds with* $C := C_{\phi^{-1}}(C_{ee}C_B + C_\phi)$.

Since Lemma 5.1 implies (5.6), the multigrid convergence is proved.

6. Application to Non-Matching Grids

In this section, we describe and analyse a multigrid method for solving systems of algebraic equations arising from a finite element method based on non-matching triangulations. The discretisation is done by the *mortar technique*, see [2, 3]. A multigrid method is presented and analysed which makes use of the general scheme presented in the previous sections.

6.1. Discrete Problem

For simplicity of presentation we restrict ourselves to the Poisson equation and assume that (2.1) is of the form: Find $u \in H_0^1(\Omega)$ such that

$$a(u, v) \equiv \int_\Omega \nabla u \nabla v dx = f(v) \equiv \int_\Omega f v dx. \tag{6.1}$$

Moreover, let $\Omega \subset \mathbb{R}^2$ be a polygonal domain and

$$\bar{\Omega} = \sum_{j\in I} \bar{\Omega}_j,$$

where Ω_j are polygons. We also assume that Ω_j form a 'triangulation', i.e., $\bar{\Omega}_i \cap \bar{\Omega}_j$ for $i \ne j$ are empty or have a common edge or vertex. In the mortar method nomenclature, this case is called *geometrically conforming*. A more general case, which is called geometrically nonconforming, will be discussed in Section 7.

Let Γ_{ij} be the open part of $\bar{\Omega}_i \cap \bar{\Omega}_j = \bar{\Gamma}_{ij}$. The union of the internal boundaries yields the *skeleton*

$$\Gamma = \bigcup_{j\in I} \partial\Omega_j \backslash \partial\Omega. \tag{6.2}$$

Let $\mathcal{T}_{0,j}$ be a coarsest triangulation in Ω_j with the mesh size[1] h_0. The ℓ-times refined triangulation is $\mathcal{T}_{\ell,j}$ with mesh parameter $h_\ell = h_0 2^{-\ell}$. The level number ℓ is assumed to range from 0 to ℓ_j. Although, in general, ℓ_j may be different in different Ω_j, we assume for simplicity that the number of levels in each Ω_j is the same, i.e., $\ell_j = \ell_{\max}$ for all $j \in I$. The standard finite element space of continuous and piecewise linear functions over the triangulation $\mathcal{T}_{\ell,j}$ is denoted by $S_{\ell,j}(\Omega_j)$. Let

$$X_\ell := X_\ell(\Omega) := \prod_{j\in I} S_{\ell,j}(\Omega_j)$$

for $\ell = 0, \ldots, \ell_{\max}$. Note that $X_\ell \not\subset H^1(\Omega)$ and functions from X_ℓ do not satisfy any continuity condition across the internal boundaries Γ_{ij}.

The nodal points of X_ℓ form the set

$$\bar{\mathcal{N}}_\ell := \bigcup_{j\in I} \bar{\mathcal{N}}_{\ell,j},$$

where $\bar{\mathcal{N}}_{\ell,j}$ is the set of nodal points of $\mathcal{T}_{\ell,j}$ on $\bar{\Omega}_j$.

Remark 6.1. The precise notation of a nodal point $p \in \bar{\mathcal{N}}_\ell$ is to be made by the pair[2] $(x_p, \Omega_{j(p)})$. An evaluation of a function u at p means values of $u|_{\Omega_{j(p)}}$ at x_p, i.e., the continuation of the function u defined on $\Omega_{j(p)}$ to $x_p \in \bar{\Omega}_{j(p)}$. In particular, at a cross point there are several nodal points $(x_p, \Omega_{j(p)})$ with identical position x_p but different $\Omega_{j(p)}$.

6.1.1. Mortar Spaces

To define suitable spaces for discretisation of (6.1) we need to impose some constraints on the jumps of functions from $X_\ell(\Omega)$ on Γ_{ij} which are called *mortar*

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[1] Although the mortar method works with different mesh sizes $h_{\ell,j}$ in the different subdomains Ω_j, we assume comparable sizes h_ℓ for all j. The reason is that the fast multigrid convergence requires similar mesh parameters.

[2] In total, we will have *three index mappings*. Here, $p \in \mathcal{N}_\ell \mapsto j(p) \in I$ maps the vertex into the index of the related subdomain. In Section 6.1, we shall introduce $m \in \mathcal{M} \mapsto i_m \in I$, where m is the master index and i_m the related subdomain index. Similarly, $m \in \mathcal{M} \mapsto j_m \in I$ maps into the subdomain index of the slave edge.

conditions. For this purpose, decompose the skeleton (6.2) into

$$\bar{\Gamma} = \bigcup_{m\in\mathcal{M}} \bar{\gamma}_m, \quad \gamma_m \neq \gamma_n \text{ if } m \neq n,\ n, m \in \mathcal{M}.$$

With each *master index* $m \in \mathcal{M}$, we associate a pair $(i, j) = (i_m, j_m) \in I \times I$ such that $\gamma_m = \Gamma_{ij}$ is an open edge of Ω_i common to Ω_j. It is called *master* (mortar) and values on γ_m will be continued from the values in Ω_i. $\Gamma_{ji} = \Gamma_{ij}$ as an open edge of Ω_j is denoted by δ_m. It is called *slave* (nonmortar), and values on δ_m will be continued from the values in Ω_j.

To be more specific, let $m \in \mathcal{M}$, $i = i_m$, $j = j_m \in I$ be as before. The (two-dimensional) triangulation $\mathcal{T}_{\ell,i}$, $\ell \in \{0, \ldots, \ell_{\max}\}$, induces a (one-dimensional) mesh $\mathcal{T}_{\ell,i} \cap \bar{\gamma}_m$ on γ_m. Similarly, one considers the mesh $\mathcal{T}_{\ell,j} \cap \bar{\delta}_m$ on δ_m. Usually, both meshes are different. The trace spaces

$$S_{\ell,m,j_m}(\delta_m) := S_{\ell,j_m}(\Omega_{j_m})|_{\delta_m} \tag{6.3}$$

are associated to these meshes on the interfaces. We note that the nodal points $p \in \bar{\mathcal{N}}_{\ell,i}$ and $q \in \bar{\mathcal{N}}_{\ell,j}$ are considered as different even if $x_p \in \mathcal{T}_{\ell,i} \cap \bar{\gamma}_m$ and $x_q \in \mathcal{T}_{\ell,j} \cap \bar{\delta}_m$ coincide since they are associated to different domains (see Remark 6.1 above).

By definition, γ_m and δ_m are open sets. The boundary $\partial\gamma_m$ ($\partial\delta_m$) consists of the two endpoints of the edge γ_m (δ_m).

Next, we introduce the space $M_{\ell,m}(\delta_m)$ as a subspace of $S_{\ell,m,j_m}(\delta_m)$ consisting of piecewise linear continuous functions on $\mathcal{T}_{\ell,j_m} \cap \bar{\delta}_m$, which are *constant* on the elements intersecting $\partial\delta_m$ (i.e., on the two end intervals of the mesh $\mathcal{T}_{\ell,j_m} \cap \bar{\delta}_m$). We say that $v_\ell \in X_\ell(\Omega)$ satisfies the *mortar condition* for $m \in \mathcal{M}$ if

$$\int_{\Gamma_{i_m,j_m}} (v_{\ell,i_m}|_{\gamma_m} - v_{\ell,j_m}|_{\delta_m})\, \psi\, ds = 0 \qquad \text{for all } \psi \in M_{\ell,m}(\delta_m). \tag{6.4}$$

The first factor in the integral is the jump of v_ℓ across Γ_{i_m,j_m} and often denoted as $[v_\ell]|_{\Gamma_{i_m,j_m}}$. Therefore the elements in $M_{\ell,m}(\delta_m)$ can be understood as Lagrange multipliers for the matching conditions.

We are now able to define the *mortar space* for discretising (6.1). Let

$$V_\ell(\Omega) := \{v_\ell \in X_\ell(\Omega) : v_\ell \text{ satisfies the mortar condition (6.4) for each } m \in \mathcal{M}\}.$$

The discrete problem is of the form: Find $u_\ell \in V_\ell(\Omega)$ such that

$$a_\ell(u_\ell, v_\ell) = f(v_\ell) \qquad \text{for all } v_\ell \in V_\ell(\Omega) \tag{6.5}$$

(cf. (2.2)), where

$$a_\ell(u_\ell, v_\ell) := \sum_{j\in I} \int_{\Omega_j} \nabla u_{\ell,j} \nabla v_{\ell,j}\, dx,$$
$$f(v_\ell) := \sum_{j\in I} \int_{\Omega_j} f v_{\ell,j}\, dx.$$

This problem has a unique solution and is stable. Moreover the error bound

$$\|u - u_\ell\|_{L_2(\Omega)} \le C h_\ell^2 \|u\|_{H^2(\Omega)} \tag{6.6}$$

holds, where C is independent of h_ℓ and u. For the proof see [7, Theorem 4.1]. The error estimate is stated there for mortar elements that are continuous at the cross points. Fortunately, the bound of the consistency error in Theorem 4.1 and the estimate in Lemma 3.4 are independent of assumptions on cross points. Inequality (6.6) is required in (2.3). A proof of the error estimate in $H^1(\Omega)$ can be found in [2] and [3] and the L_2 estimate is also given in [1] without a proof.

Our goal is to design and analyse a multigrid method for solving the linear system (6.9) from below corresponding to (6.5). The next three subsections will be a preparation for that.

6.1.2. Matrix Form

We rewrite the problem (6.5) in a matrix form using basis functions of $V_\ell(\Omega)$, i.e., basis functions satisfying the mortar condition. We emphasise that we do not require continuity at the cross points for the functions in $V_\ell(\Omega)$.

Assume that $v_\ell \in X_\ell$ satisfies the mortar condition (6.4). Note that $v_{\ell,j_m}|_{\delta_m}$ can be computed at interior nodal points of $\mathcal{T}_{\ell,j_m} \cap \delta_m$ from the values of $v_{\ell,i_m}|_{\gamma_m}$ on $\bar\gamma_m$ and the two values of v_{ℓ,j_m} at the endpoints of δ_m. The restriction of a function v on δ_m to the endpoints of δ_m is denoted by $Tr_m\, v$. We have $v_{\ell,j_m}|_{\delta_m} := \Pi_{\ell,m}(v_{\ell,i_m}, Tr_m\, v_{\ell,j_m})$, where $\Pi_{\ell,m}$ is a mapping from $L_2(\Gamma_{i_m,j_m}) \times Tr_m(\delta_m)$ onto $S_{\ell,m,j_m}(\delta_m)$ (see (6.3)) and given by the solution $w \in S_{\ell,m,j_m}(\delta_m)$ of

$$\int_{\Gamma_{i_m,j_m}} (w - v_{\ell,i_m})\, \psi\, ds = 0 \qquad \text{for all } \psi \in M_{\ell,m}(\delta_m), \tag{6.7}$$
$$Tr_m\, w = Tr_m\, v_{\ell,j_m}. \tag{6.8}$$

To find $\Pi_{\ell,m}(v_{\ell,i_m}, Tr_m\, v_{\ell,j_m})$, we only need to solve a system with a tridiagonal mass matrix.

Let $\mathcal{N}_{\ell,j}$ and $\mathcal{N}_\ell$ be the set of nodal points of $\bar{\mathcal{N}}_{\ell,j}$ and $\bar{\mathcal{N}}_\ell$, respectively, except those which belong to interiors of the slave edges, i.e.,

$$\mathcal{N}_{\ell,j} := \bar{\mathcal{N}}_{\ell,j} \setminus \bigcup_{m \in \mathcal{M}} \delta_m, \qquad \mathcal{N}_\ell := \bigcup_{j \in I} \mathcal{N}_{\ell,j}.$$

In the following, we describe the *Lagrange basis* $\{b_{\ell,p} : p \in \mathcal{N}_\ell\}$ of $V_\ell(\Omega)$. It is uniquely defined by $b_{\ell,p} \in V_\ell(\Omega)$ and $b_{\ell,p}(x_q) = \delta_{pq}$ (Kronecker symbol δ). We discuss the structure of these basis functions in detail.

Recalling Remark 6.1, we associate with each $p = (x_p, \Omega_{j(p)}) \in \mathcal{N}_\ell$ a basis function $b_{\ell,p} \in V_\ell(\Omega)$ and distinguish three cases. For the standard nodal basis function in X_ℓ corresponding to the vertex $p \in \bar{\mathcal{N}}_{\ell,j(p)}$, the subdomain $\Omega_{j(p)}$, and the triangulation $\mathcal{T}_{\ell,j(p)}$, we use the notation $\varphi_{\ell,p}$. Concerning the notations $j(p)$, i_m and j_m we refer to Footnote 2.

Case I (interior nodal point) For $x_p \in \Omega_{j(p)}$, $b_{\ell,p}|_{\Omega_{j(p)}} = \varphi_{\ell,p}$ is the standard nodal basis function on $\bar{\Omega}_{j(p)}$ and $b_{\ell,p} = 0$ on $\Omega \setminus \bar{\Omega}_{j(p)}$. In this case, $b_{\ell,p}$ is continuous on the whole domain Ω.

Case II (nodal point on interior of an interior boundary) Let $m \in \mathcal{M}$ and $x_p \in \gamma_m$. Hence, $i_m = j(p)$. Obviously, $p \in \mathcal{N}_{\ell,i_m}$, i.e., the nodal point lies in the *interior of a master edge* γ_m of Ω_{i_m}. Then

$$b_{\ell,p} = \begin{cases} \varphi_{\ell,p} & \text{on } \bar{\Omega}_{i_m}, \\ \Pi_{\ell,m}(\varphi_{\ell,p}, 0) & \text{on } \bar{\delta}_m \subset \partial\Omega_{j_m}, \\ 0 & \text{at all other nodal points in } \bar{\Omega} \setminus \left(\bar{\Omega}_{i_m} \cup \bar{\delta}_m\right). \end{cases}$$

The support of $b_{\ell,p}$ consists of all triangles from $\mathcal{T}_{\ell,j_m}$ which touch the (open) slave edge δ_m and the triangles from $\mathcal{T}_{\ell,i_m}$ which touch x_p. Note that $b_{\ell,p}$ is discontinuous at Γ_{i_m,j_m}.

Case III (cross point as nodal point) A cross point may belong to several $\bar{\Omega}_i$ and to various $\partial\gamma_m$ or $\partial\delta_m$. Let $p = (x_p, \Omega_{j(p)})$. The subdomain $\Omega_{j(p)}$ has two edges joining at x_p. Each of these edges may be mortar edges γ_m (then $i_m = j(p)$) or nonmortar edges δ_m (then $j_m = j(p)$). We define $b_{\ell,p}$ by $b_{\ell,p}(p) = 1$ and

$$b_{\ell,p} = \begin{cases} \varphi_{\ell,p} & \text{on } \gamma_m \subset \partial\Omega_{i_m} \text{ if } x_p \in \partial\gamma_m,\ i_m = j(p), \\ \Pi_{\ell,m}(0, Tr_m\, \varphi_{\ell,p}) & \text{on } \delta_m \subset \partial\Omega_{i_m} \text{ if } x_p \in \partial\delta_m,\ i_m = j(p), \\ \Pi_{\ell,m}(\varphi_{\ell,p}, 0) & \text{on } \delta_m \subset \partial\Omega_{j_m} \text{ if } x_p \in \partial\delta_m,\ j_m = j(p), \\ 0 & \text{at all other nodal points in } \bar{\Omega}. \end{cases}$$

The first two rows correspond to mortar/nonmortar edges associated to the subdomain $\Omega_{j(p)}$, while in the third row δ_m is a nonmortar belonging to a neighbouring subdomain.

Using the basis functions above, we have $V_\ell(\Omega) = span\{b_{\ell,p} : p \in \mathcal{N}_\ell\}$. Problem (6.5) takes the matrix form

$$A_\ell \, \underline{u}_\ell = \underline{f}_\ell. \tag{6.9}$$

The matrix A_ℓ is symmetric and positive definite. Moreover it follows from the usual inverse estimates for X_ℓ and $V_\ell \subset X_\ell$ that the eigenvalues $\lambda_i(A_\ell)$ of A_ℓ satisfy the inequalities

$$C_0 \le \lambda_i(A_\ell) \le C_1 h_\ell^{-2} \tag{6.10}$$

where C_0 and C_1 are independent of h_ℓ.

6.1.3. Nonconformity

In Section 6.1.2, we have constructed a family of finite element spaces $V_\ell(\Omega)$ ($\ell = 0, 1, \ldots, \ell_{\max}$) which satisfy the mortar condition (6.4) of the respective level. We have started with nested spaces

$$X_0(\Omega) \subset X_1(\Omega) \subset \ldots \subset X_\ell(\Omega) \subset \prod L_2(\Omega_j).$$

However, the subspaces $V_\ell(\Omega)$ are not nested. The reason is that a function $u_{\ell-1}|_{\Omega_{i_m}} = u_\ell|_{\Omega_{i_m}}$ belonging to $S_{\ell-1,i_m}(\Omega_{i_m}) \subset S_{\ell,i_m}(\Omega_{i_m})$ yields different values $u_{\ell-1}|_{\Omega_{j_m}} \neq u_\ell|_{\Omega_{j_m}}$ on the slave side since the mortar condition for level $\ell-1$ does not imply the mortar condition for level ℓ, in general. Hence,

$$V_0(\Omega) \not\subset V_1(\Omega) \not\subset \ldots \not\subset V_\ell(\Omega).$$

6.2. Estimates

The analysis of the multigrid method reduces to the analysis of the two-grid one (cf. Subsection 5.2). Therefore the facts needed for that analysis are formulated for the levels ℓ and $\ell - 1$.

6.2.1. L_2-Stability of the Mortar Projection

It is known that the mortar projection $\Pi_{\ell,m}$ (cf. (6.7) and (6.8)) is L_2-stable.

Lemma 6.2. *Assume that $v_\ell \in X_\ell$ satisfies the mortar condition on $\delta_m = \gamma_m = \Gamma_{i_m,j_m}$. If $Tr_m\, v_{\ell,j_m} = 0$, then*

$$\|v_{\ell,j_m}\|_{L_2(\delta_m)} \le C \|v_{\ell,i_m}\|_{L_2(\gamma_m)} \tag{6.11}$$

where the constant C is independent of h_ℓ.

The proof of this lemma is given in [2, Lemma 2.1] for the 3-D case with uniform triangulation and in [6] for general ones. Its simplification for the 2-D case can be found, e.g., in [7, Lemma 3.1] and [10, the proof of Lemma 1].

Remark 6.3. We note that we will apply a slight generalisation of the lemma above which is also covered by the literature. Specifically, inequality (6.11) remains true if v_{ℓ,i_m} is an arbitrary L_2-function on γ_m.

6.2.2. ℓ_2 Inequalities

In the following, we denote by $\ell_2(\mathcal{N})$ the space of tuples over an index set $\mathcal{N}$. Here, $\mathcal{N}$ takes the values $\bar{\mathcal{N}}_{\ell,j}$, $\bar{\mathcal{N}}_\ell$, and $\mathcal{N}_{\ell,j}$, $\mathcal{N}_\ell$. The appropriate scaling of the ℓ_2-norms regarding the equivalence (2.6) is

$$\|\underline{v}_\ell\|_{\ell_2(\mathcal{N})} := \sqrt{\sum_{p\in\mathcal{N}} h_\ell^2\,|\underline{v}_\ell(p)|^2} \qquad \text{for } \mathcal{N} = \mathcal{N}_{\ell,j}, \bar{\mathcal{N}}_{\ell,j}, \mathcal{N}_\ell, \text{ and } \bar{\mathcal{N}}_\ell.$$

Here, $\underline{v}_\ell(p) = v_\ell(x_p)$ is the notation for the p-component of the vector $\underline{v}_\ell$ according to Remark 6.1.

The well-known equivalence between ℓ_2 and L_2 norms is first stated for the space X_ℓ. Note that we have assumed quasi-uniformity of the triangulations in each Ω_j, $j \in I$.

Lemma 6.4. *(a) Let $\bar{\phi}_{\ell,j}$ be the isomorphism from $\ell_2(\bar{\mathcal{N}}_{\ell,j})$ onto $S_{\ell,j}(\Omega_j)$ (cf. (2.5)). Then there is a constant C independent of j and the mesh size h_ℓ such that*

$$\frac{1}{C}\|\underline{v}_{\ell,j}\|_{\ell_2(\bar{\mathcal{N}}_{\ell,j})} \le \|\bar{\phi}_{\ell,j}(\underline{v}_{l,j})\|_{L_2(\Omega_j)} \le C\|\underline{v}_{\ell,j}\|_{\ell_2(\bar{\mathcal{N}}_{\ell,j})} \qquad \text{for } \underline{v}_{\ell,j} \in \ell_2(\bar{\mathcal{N}}_{\ell,j}).$$

(b) Similarly, for $\bar{\phi}_\ell$ being the isomorphism between $\ell_2(\bar{\mathcal{N}}_\ell)$ and $X_\ell(\Omega)$, we have

$$\frac{1}{C}\|\underline{v}_\ell\|_{\ell_2(\bar{\mathcal{N}}_\ell)} \le \|\bar{\phi}_\ell(\underline{v}_\ell)\|_{L_2(\Omega)} \le C\|\underline{v}_\ell\|_{\ell_2(\bar{\mathcal{N}}_\ell)} \qquad \text{for } \underline{v}_\ell \in \ell_2(\bar{\mathcal{N}}_\ell). \tag{6.12}$$

Since $\mathcal{N}_{\ell,j} \subseteq \bar{\mathcal{N}}_{\ell,j}$, the following estimate is trivial:

$$\|\underline{v}_\ell\|_{\ell_2(\mathcal{N}_\ell)} \le \|\underline{v}_\ell\|_{\ell_2(\bar{\mathcal{N}}_\ell)}. \tag{6.13}$$

The opposite inequality is obviously not valid for coefficient vectors $\underline{v}_\ell \in \ell_2(\bar{\mathcal{N}}_\ell)$ belonging $v_\ell \in X_\ell(\Omega)$. However, it is true if $v_\ell \in V_\ell(\Omega)$, as stated in

Lemma 6.5. *Let $\phi_\ell : \underline{v}_\ell \in \ell_2(\mathcal{N}_\ell) \mapsto v_\ell = \sum_{p\in\mathcal{N}_\ell} \underline{v}_\ell(p)\, b_{\ell,p} \in V_\ell(\Omega)$ with basis functions from Section 6.1.2 be the isomorphism from $\ell_2(\mathcal{N}_\ell)$ onto $V_\ell(\Omega)$.*

Let $\underline{x}_\ell := \bar{\phi}_\ell^{-1} v_\ell \in \ell_2(\bar{\mathcal{N}}_\ell)$ *be the coefficient vector for* $v_\ell \in X_\ell(\Omega) \supset V_\ell(\Omega)$. *Then we have* $\underline{x}_\ell(p) = \underline{v}_\ell(p)$ *for all vertices* $p \in \mathcal{N}_\ell$ *and*

$$\|\underline{x}_\ell\|_{\ell_2(\bar{\mathcal{N}}_\ell)} \leq C \|\underline{v}_\ell\|_{\ell_2(\mathcal{N}_\ell)}. \tag{6.14}$$

Proof: Given a slave edge $\delta := \delta_m \subset \bar{\Omega}_{j_m}$ and the corresponding master edge $\gamma := \gamma_m \subset \bar{\Omega}_{i_m}$, let $A := A_m$ and $B := B_m$ be the endpoints of δ. The proof of (6.14) reduces to show that for each $\delta = \delta_m \subset \Gamma$, $m \in \mathcal{M}$,

$$\|\underline{x}_\ell\|_{\ell_2(\bar{\mathcal{N}}_\ell \cap \delta)} \leq C \|\underline{v}_\ell\|_{\ell_2(\mathcal{N}_\ell \cap (\bar{\gamma} \cup A \cup B))}. \tag{6.15}$$

Here $\bar{\mathcal{N}}_\ell \cap \delta$ is the set of interior nodal points of δ which by the definition of $\Pi_{\ell,m}$ depend on $\mathcal{N}_\ell \cap (\bar{\gamma} \cup A \cup B))$. Now define $w_\ell \in L_2(\Gamma_{i_m, j_m})$ by $w_\ell \in S_{\ell,m,j_m}$ and

$$w_\ell = \begin{cases} v_\ell & \text{at the nodal points } A \text{ and } B, \\ 0 & \text{at the nodal points of } \delta. \end{cases}$$

Roughly speaking w_ℓ is obtained from $v_\ell|_{\bar{\delta}}$ by the restriction to the values on $\partial\delta$. Note that $v_{\ell,i_m} - w_\ell$ and $v_{\ell,j_m} - w_\ell$ also satisfy the mortar condition (6.4). With this we have $(v_{\ell,j_m} - w_\ell)|_\delta = \Pi_{m,\ell}(v_{\ell,i_m} - w_\ell, 0)$. It follows from Lemma 6.2 that

$$\|v_\ell\|_{L_2(\delta)} \leq \|w_\ell\|_{L_2(\delta)} + C(\|v_\ell\|_{L_2(\gamma)} + \|w_\ell\|_{L_2(\gamma)}).$$

After rewriting this in terms of the coefficients, the proof of (6.15) is complete. □

Proposition 6.6. *The isomorphism* ϕ_ℓ *from* $\ell_2(\mathcal{N}_\ell)$ *onto* $V_\ell(\Omega)$ *satisfies*

$$\frac{1}{C} \|\underline{v}_\ell\|_{\ell_2(\mathcal{N}_\ell)} \leq \|\phi_\ell(\underline{v}_\ell)\|_{L_2(\Omega)} \leq C \|\underline{v}_\ell\|_{\ell_2(\mathcal{N}_\ell)} \quad \textit{for all } \underline{v}_\ell \in \ell_2(\mathcal{N}_\ell).$$

Proof: Set $v_\ell := \phi_\ell(\underline{v}_\ell)$ and extend the vector $\underline{v}_\ell$ by the components of $\underline{x}_\ell := \bar{\phi}_\ell^{-1} v_\ell$ at $p \in \bar{\mathcal{N}}_\ell \backslash \mathcal{N}_\ell$. Combination of (6.13) and (6.12) yields

$$\|\underline{v}_\ell\|_{\ell_2(\mathcal{N}_\ell)} \leq C \; \|v_\ell\|_{L_2(\Omega)} \, .$$

From (6.12d (6.14) we obtain the final part

$$\|v_\ell\|_{L_2(\Omega)} \leq C \|\underline{v}_\ell\|_{\ell_2(\bar{\mathcal{N}}_\ell)} \leq C^2 \|\underline{v}_\ell\|_{\ell_2(\mathcal{N}_\ell)}.$$

□

6.3. Choice of Σ, *S*, σ, *and* π

In the diagram below, $V_{\ell-1}$ and V_ℓ are the mortar spaces introduced in Section 6.1.1. The coefficient spaces $\mathcal{U}_j = \ell_2(\mathcal{N}_j)$, $(j = \ell - 1, \ell)$, together with their isomorphisms ϕ_j have already been specified.

$$\begin{array}{ccccc}
 & \Sigma = X_\ell & \xrightarrow{\sigma} & S = \mathcal{U}_\ell = \ell_2(\mathcal{N}_\ell) & \\
\text{inclusion}\uparrow & & \nwarrow & \downarrow \pi = \phi_\ell & \\
 & V_{\ell-1} & \xrightarrow[\iota]{} & V_\ell & \\
\phi_{\ell-1}\uparrow & & & \uparrow \phi_\ell & \\
 & \mathcal{U}_{\ell-1} & \xrightarrow[p]{} & \mathcal{U}_\ell &
\end{array}$$

We choose $\Sigma := X_\ell$ equipped with the norm of $L_2(\Omega)$. By construction $V_\ell = V_\ell(\Omega) \subset X_\ell$ holds. Since $X_{\ell-1} \subset X_\ell$, also $V_{\ell-1} \subset X_\ell$. Hence, the sum $V_\ell + V_{\ell-1}$ is contained in X_ℓ.

The space S is chosen as $S := \mathcal{U}_\ell = \ell_2(\mathcal{N}_\ell)$. The obvious choice for π is $\pi := \phi_\ell$. Injectivity and boundedness of π is given by Proposition 6.6.

The essential part is the choice of the mapping $\sigma : X_\ell \to \ell_2(\mathcal{N}_\ell)$. Given a patch-wise continuous function v from X_ℓ and indices $p = (x_p, \Omega_{i(p)}) \in \mathcal{N}_\ell$, we set

$$(\sigma v)\,(p) := v(x_p)$$

according to Remark 6.1. Then $\sigma v = ((\sigma v)\,(p))_{p \in \mathcal{N}_\ell}$ is the coefficient vector in $\ell_2(\mathcal{N}_\ell)$.

Remark 6.7. $\sigma : X_\ell \to \ell_2(\mathcal{N}_\ell)$ is bounded uniformly in ℓ:

$$\|\sigma v_\ell\|_{\ell_2(\mathcal{N}_\ell)} \leq C \|v_\ell\|_{L_2(\Omega)}.$$

Proof: Set $\underline{w}_\ell := \sigma v_\ell$. The combination of (6.13) and (6.12) yields

$$\|\underline{w}_\ell\|_{\ell_2(\mathcal{N}_\ell)} = \|\underline{v}_\ell\|_{\ell_2(\mathcal{N}_\ell)} \leq C \|v_\ell\|_{L_2(\Omega)}.$$

□

We note that the opposite inequality $\|v_\ell\|_{L_2(\Omega)} \leq C \|\underline{v}_\ell\|_{\ell_2(\mathcal{N}_\ell)}$ is not valid for all $v_\ell \in V_\ell + V_{\ell-1} \subset \Sigma = X_\ell$. There are nontrivial functions u_ℓ in $V_\ell + V_{\ell-1}$ with $\sigma(u_\ell) = 0$. For its construction take a function $v_{\ell-1}$ with support in Ω_i and set $v_\ell := v_{\ell-1}$. Extend $v_{\ell-1}$ and v_ℓ to the slave side according to the different mortar conditions of level ℓ and $\ell - 1$, respectively. Then the function $u_\ell := v_\ell - v_{\ell-1}$ vanishes on Ω_i, but it is nonzero on the slave side. After a slight modification at nodal points neighboured to the slave nodes, one obtains zero coefficients for all $p \in \mathcal{N}_\ell$, but $u_\ell \neq 0$. Thus, σ is not injective.

Remark 6.8. $\iota = \phi_\ell \circ \sigma$ is a projection onto V_ℓ.

Proof: Given $v_\ell \in V_\ell$, $\underline{v}_\ell := \sigma v_\ell$ are the values at the nodal points of $\mathcal{N}_\ell$. Because of the Lagrange basis property, $\phi_\ell \underline{v}_\ell$ recovers the function $v_\ell \in V_\ell$. $\square$

So far, we have ensured the conditions (3.2)–(3.4) required in Section 3.2.

Finally, we observe that inequality (5.1) follows from

$$|a_\ell(u_\ell, v_\ell)| \leq C \, \|u_\ell\|_{H^1_{broken}} \, \|v_\ell\|_{H^1_{broken}}$$

with $\|v_\ell\|^2_{H^1_{broken}} := \sum_{j \in I} \|v_\ell|_{\Omega_j}\|^2_{H^1(\Omega_j)}$ and the standard inverse inequality

$$\|v_\ell|_{\Omega_j}\|_{H^1(\Omega_j)} \leq C h_\ell^{-1} \|v_\ell|_{\Omega_j}\|_{L_2(\Omega_j)} \quad \text{for } v_\ell \in S_{\ell,j}(\Omega_j).$$

Hence, if the underlying second order boundary value problem satisfies the solvability and regularity conditions, all necessary requirements for the two- and multigrid convergence are satisfied.

7. Geometrically Nonconforming Case

In this section, we discuss a discretisation of the problem (6.1) on non-matching triangulations in the case when the subdomains Ω_j of the decomposition $\bar{\Omega} = \bigcup_{j \in I} \bar{\Omega}_j$ do not form a 'triangulation' of Ω. In our case the Ω_j's are assumed to be polygons, and this means that a vertex of Ω_j is not necessarily a vertex of its neighbours Ω_i. This case is called the *geometrically nonconforming* case in the mortar method, and a typical situation is depicted in Fig 1.

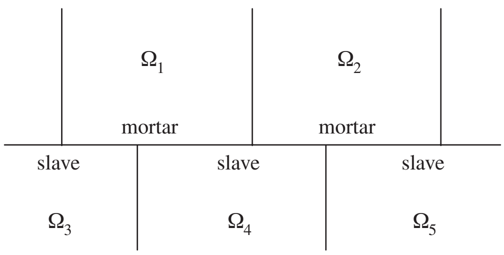

Figure 1. Detail of a geometrically nonconforming mesh

We first formulate a discrete problem for (6.1) and then extend our treatment of the geometrically conforming case to this one.

7.1. Discrete Problem

To formulate the discrete problem, we will adapt some of our previous notations and introduce some new ones. Let here $\Gamma_{j,k}$, $k = 1, \ldots, K_j$, denote the edges of Ω_j and K_j be its number. Let $\Gamma = \bigcup_{j\in I} \partial\Omega_j \backslash \partial\Omega$ be as in (6.2). We select a family of edges, say $\{\gamma_k,\ k \in K\}$, from the set of all edges $\{\Gamma_{j,k} :\ j \in I,\ 0 \le k \le K_j\}$ satisfying the condition

$$\bigcup_{k\in K} \bar{\gamma}_k = \bar{\Gamma}, \quad \gamma_k \cap \gamma_{k'} = \emptyset \text{ if } k \neq k'. \tag{7.1}$$

These edges are called masters (mortars). The edges $\Gamma_{j,k}$ which are not masters are called slaves (nonmortars) and denoted by δ_m, $m \in \mathcal{M}$. Note that the nonmortars $\{\delta_m\}$ satisfy a condition similar to (7.1):

$$\bigcup_{m\in \mathcal{M}} \bar{\delta}_m = \bar{\Gamma}, \quad \delta_m \cap \delta_{m'} = \emptyset \text{ if } m \neq m'.$$

We adapt the notation δ_m (γ_k) as an edge of Ω_{j_m} (Ω_{i_k}). A mortar space $M_{\ell,m}(\delta_m)$ is defined on each δ_m in the same way as in Section 6.

We say that a function $v_\ell \in X_\ell(\Omega)$ satisfies the mortar condition on an edge δ_m if

$$\int_{\delta_m} [v_\ell]\, \psi\, ds = 0 \qquad \text{for all } \psi \in M_{\ell,m}(\delta_m), \tag{7.2}$$

where $[v_\ell]$ is the jump of v_ℓ. The discretisation space $V_\ell(\Omega)$ for (6.1) is defined as the subspace of $X_\ell(\Omega)$ of functions satisfying the mortar conditions (7.2) for all $m \in \mathcal{M}$.

We are now able to formulate the discrete problem. *Find $u_\ell \in V_\ell(\Omega)$ such that*

$$a_\ell(u_\ell, v_\ell) = f(v_\ell) \qquad \text{for all } v_\ell \in V_\ell. \tag{7.3}$$

The problem has a unique solution, and the error is bounded by

$$\|u_\ell - u\|_{L_2(\Omega)} \le C h_\ell^2 \sum_{j\in I} \|u\|_{H^2(\Omega)},$$

where u_ℓ and u are the solutions of (7.3) and (6.1), respectively. The proof of the error estimates proceeds as in the geometrically conforming case. Specifically the extension of the stability results is based on Remark 6.3. The estimates of the consistency errors are local in nature and carry over without changes.

Our goal is to design and analyse the multigrid method for solving (7.3) using the previous scheme.

7.2. Matrix Form

We rewrite (7.3) in the matrix form using that

$$V_\ell = span\{b_{\ell,p} :\ p \in \mathcal{N}_\ell\}.$$

The basis functions $b_{\ell,p}$ are defined again by the Lagrange basis property similar to those in Section 6. For $p \in \mathcal{N}_\ell$ with x_p lying in the interior of Ω_{j_p}, $b_{\ell,p}$ is the same as in Section 6. For $x_p \in \gamma_m$, where γ_m is a master edge in $\bar{\Omega}_{j(p)}$, $b_{\ell,p}$ is defined by

$$b_{\ell,p} = \begin{cases} \varphi_{\ell,p} & \text{in } \bar{\Omega}_{j(p)} \\ \Pi_{\ell,n}(\varphi_{\ell,p}, 0) & \text{on } \delta_n \end{cases},$$

where δ_n, the slave edge of Ω_{j_n}, is such that $\emptyset \neq \delta_n \cap \gamma_m \subset \text{supp}(\varphi_{\ell,p})$; on the remaining vertices in $\bar{\Omega}_{j_n}$ and the remaining substructures $b_{\ell,p}(x)$ is extended by zero. For x_p which are vertices of Ω_{i_p}, $b_{\ell,p}$ are defined in the same way as in Section 6. Here, the extension by the mortar projection is understood as in (7.2).

Using $b_{\ell,p}$, we obtain

$$A_\ell\, \underline{u}_\ell = \underline{f}_\ell. \tag{7.4}$$

The matrix A_ℓ is symmetric and positive definite and its eigenvalues satisfy also the inequalities (6.10).

7.3. Two-Grid and Multigrid Method

We comment here on the two-grid method for (7.3) (or (7.4)). This is sufficient to analyse the multigrid method.

Since ℓ is a variable level number, the space $V_{\ell-1}$ and the mortar projection $\Pi_{\ell-1,m}$ are defined as well. We first formulate the counterpart of Lemma 6.2 for the geometrically nonconforming case. Let δ_m be a slave edge of Ω_{j_m} and let $v_{\ell,j_m} \in S_{\ell,j_m}(\Omega_{j_m})$ be the components of $v_\ell = \{v_{\ell,j}\} \in V_\ell(\Omega)$. By the definition

$$v_{\ell,j_m|\delta_m} = \Pi_{\ell,m}(\tilde{v}_\ell, Tr_m v_{\ell,j_m}) \tag{7.5}$$

where $\tilde{v}_{\ell|\delta_m} = \sum_k v_{\ell,i_k|\delta_m \cap \gamma_k}$ and the sum is taken over v_{i_k} defined on $\bar{\Omega}_{i_k}$ involving γ_k with $\delta_m \cap \gamma_k \neq \emptyset$.

Lemma 7.1. *For $v_\ell = \{v_{\ell,j}\}_{j\in I} \in V_\ell$ on $\delta_m \subset \bar{\Omega}_{j_m}$ with $Tr_m v_{j_m} = 0$ holds*

$$\|v_{\ell,j_m}\|^2_{L_2(\delta_m)} \le C \sum_k \|v_{\ell,i_k}\|^2_{L_2(\gamma_k\cap\delta_m)},$$

where v_{ℓ,j_m} is defined by (7.5) and C is a constant independent of h_{ℓ,i_k}.

For a proof of this lemma, see [2, Lemma 2.1].

Using this lemma, we check that Lemmata 6.2, 6.4 and 6.5 from Section 6 are also valid for the discussed case. Defining σ, π and $\iota = \sigma \circ \pi$ as in the previous section, we check that π is injective and bounded, σ is bounded and ι is a projection. Thus the two-grid method for (7.3) (and (7.4)) is uniformly convergent with respect to h_ℓ.

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