Basic Motivation

- reasoning on the basis of inconsistent information
- use a paraconsistent logic
- however, they are weak, if monotonic
- why?
  - disjunctive syllogism (If $A$ and $\neg A \vee B$ then $B$) doesn’t hold
  - do you see why?

Outline

1. Inconsistency-Adaptive Logics

2. Prioritising Adaptive Logics

What to do?

- strengthen paraconsistent logics adaptively
- if we assume $A \land \neg A$ not to hold we get that
  - $A$ and $\neg A \lor B$ implies $B$
  - $\neg A$ and $A \lor B$ implies $B$
So . . .

- we need to first settle for a paraconsistent lower limit logic
- a good candidate for the logical form of abnormalities is ∼A ∧ A

Enters: CLuN

CLuN is defined by Modus Ponens (MP) and the following axiom schemata:

1. \((A \supset B) \supset (A \supset (B \supset A))\)
2. \((A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))\)
3. \(((A \supset B) \supset A) \supset A\)
4. \((A \land B) \supset A\)
5. \((A \land B) \supset B\)
6. \((A \land B) \supset (A \land B)\)
7. \(A \supset (A \lor B)\)
8. \((A \lor B) \supset (A \lor B)\)
9. \((A \lor B) \supset (A \lor B)\)
10. \((A \lor B) \supset (A \lor B)\)
11. \((A \supset \neg A) \supset \neg A\)
12. \((A \supset \neg A) \supset \neg A\)
13. \((A \supset \neg A) \supset \neg A\)
14. \((A \supset \neg A) \supset \neg A\)
15. \(A \lor \neg A\)
16. \(A \lor \neg A\)

The semantics of CLuN

- take an assignment \(v : W \rightarrow \{0, 1\}\) where \(W\) is the set of wffs
- define \(v_M\) as follows:
  - where \(A \in A\): \(v_M(A) = v(A)\) (where \(A\) is the set of atoms)
  - \(v_M(A \lor B) = \max(v_M(A), v_M(B))\)
  - \(v_M(A \land B) = \min(v_M(A), v_M(B))\)
  - \(v_M(\neg A) = 1 - v_M(A)\)
  - and either define \(A \supset B\) by \(\neg A \lor B\) or use
    \(v_M(A \supset B) = \max(1 - v_M(A), v_M(B))\)

CLuN is weak . . .

- no double negation elimination/introduction, no De Morgan Rules:
  - \(A \not\in\text{CLuN} \quad \neg \neg A\)
  - \(\neg \neg A \not\in\text{CLuN} \quad A\)
  - \(\neg (A \lor B) \not\in\text{CLuN} \quad \neg A \land \neg B\)
  - \(\neg A \land \neg B \not\in\text{CLuN} \quad \neg (A \lor B)\)
  - etc.
Now, let's go adaptive

Note that:

- $A, \sim A \lor B \vdash_{CLuN} B \lor (A \land \sim A)$
- $A \vdash_{CLuN} \sim \sim A \lor (A \land \sim A)$
- $\sim \sim A \vdash_{CLuN} A \lor (\sim A \land \sim \sim A)$
- $\sim(A \lor B) \vdash_{CLuN} (\sim A \land \sim B) \lor ((A \lor B) \land \sim (A \lor B))$
- etc.

Thus:

- $A, \sim A \lor B \vdash_{CLuN^s} B$
- $A \vdash_{CLuN^s} \sim \sim A$
- $\sim \sim A \vdash_{CLuN^s} A$
- $\sim(A \lor B) \vdash_{CLuN^s} (\sim A \land \sim B)$
- etc.

What about strengthening CLuN? Enters: CLuNs

{CLuNs} is {CLuN} plus De Morgan and plus double negation introduction and elimination

The (2-valued) semantics of CLuNs

- take an assignment $v : \mathcal{L} \rightarrow \{0, 1\}$ where $\mathcal{L}$ is the set of literals
- define $v_M$ as follows:
  - where $A \in \mathcal{A}$: $v_M(A) = v(A)$
  - where $A \in \mathcal{A}$: $v_M(\sim A) = \max(1 - v_M(A), v(\sim A))$
  - $v_M(A \lor B) = \max(v_M(A), v_M(B))$
  - $v_M(A \land B) = \min(v_M(A), v_M(B))$
  - $v_M(\sim \sim A) = v_M(A)$
  - $v_M(\sim(A \lor B)) = v_M(\sim A \land \sim B)$
  - $v_M(\sim(A \land B)) = v_M(\sim A \lor \sim B)$
  - and for CLuNs also:
    - $v_M(\sim A) = 1 - v_M(A)$
    - and either define $A \supset B$ by $\sim A \lor B$ or use $v_M(A \supset B) = \max(1 - v_M(A), v_M(B))$
## The Flip/Flop problem

- Suppose we use the same unrestricted form of abnormalities $A \land \neg A$ as for CLuN.
- Do you see a problem if we use this logic and reason, for instance, with $\Gamma = \{p, \neg p \lor q, r, \neg r\}$?

## What’s the cure?

- Restrict the form of abnormalities to atoms!

## Some abnormalities count more . . .

- Idea: structure the set of abnormalities!
- Instead of $\Omega$ we use $\langle \Omega_1, \Omega_2, \ldots \rangle$ where each $\Omega_i$ is specified by a logical form.

### Example

$\odot_i A \land \neg A$ (rate the trustworthiness of your evidence: the lower $i$ the higher the trustworthiness of the evidence)

## Semantics based on this idea

- Start with the set $M_{LLL}(\Gamma)$ of all LLL-models of $\Gamma$.
- Now select all $\Omega_1$-minimally abnormal models out of these: $M_1(\Gamma)$.
- Then select all $\Omega_2$-minimally abnormal models out of $M_1(\Gamma)$ denoted by $M_2(\Gamma)$.
- Go on like this.
- The set of all selected models $M_\rho(\Gamma)$ is given by $\bigcap_{i \geq 1} M_i(\Gamma)$.
- The consequence relation is then defined as usual: $\Gamma \models A$ iff for all $M \in M_\rho(\Gamma)$, $M \models A$. 

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**Christian Straßer**

Inconsistency-Adaptive Logics

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Christian Straßer

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An alternative semantics

Where $\Delta \subseteq \bigcup_{i \geq 1} \Omega_i$, we define $\Delta_i = \Delta \cap \Omega_i$.

Lexicographic Ordering

$\Delta \prec_{\text{lex}} \Theta$ iff there is an $i \geq 1$ such that
- for all $j \leq i$: $\Delta_j = \Theta_j$
- $\Delta_i \subset \Theta_i$.

Order Models according to abnormal parts and lex-order

Let $M \sqsubseteq M'$ iff $\text{Ab}(M) \prec_{\text{lex}} \text{Ab}(M')$.

$\Gamma \Vdash A$ iff for all $\sqsubseteq$-minimal models $M$ in $\mathcal{M}_{\text{LLL}}(\Gamma)$, $M \models A$.

The two semantics are identical

We have to show that $M \in \mathcal{M}_p(\Gamma)$ iff $M$ is $\sqsubseteq$-minimal in $\mathcal{M}(\Gamma)$.

$(\Rightarrow)$

Suppose $M \in \mathcal{M}_p(\Gamma)$.

Assume for a contradiction that there is a $M' \in \mathcal{M}(\Gamma)$ for which $\text{Ab}(M') \prec_{\text{lex}} \text{Ab}(M)$.

Hence, there is a minimal $i$ for which $\text{Ab}(M')_i \subset \text{Ab}(M)_i$ and $\text{Ab}(M')_j = \text{Ab}(M)_j$ for all $j < i$.

Hence, both $M, M' \in \mathcal{M}_i(\Gamma)$ for all $j < i$.

However, $\text{Ab}(M')_i \subset \text{Ab}(M)_i$ contradicts the fact that $M \in \mathcal{M}_i(\Gamma)$.

$(\Leftarrow)$

Similar (try it!).

A simple example

Take $\Gamma = \{\circ_1 p, \circ_2 q, \circ_3 r, \neg p \lor \neg q \lor \neg r\}$

Here are some models

<table>
<thead>
<tr>
<th>model</th>
<th>$\circ_1 p \land \neg p$</th>
<th>$\circ_2 q \land \neg q$</th>
<th>$\circ_3 r \land \neg r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_1$</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$M_2$</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$M_3$</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$M_4$</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$M_5$</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Can you determine $\mathcal{M}_i(\Gamma)$ for $i \in \{1, 2, 3\}$?

What follows in the example?

What about the proof theory?

- basic idea: similar to minimal abnormality strategy
- just instead of $\Phi_s(\Gamma)$ use a different set: $\Phi^\text{lex}_s(\Gamma)$:

$\Phi^\text{lex}_s(\Gamma)$ is the set of all choice sets of $\Sigma(\Gamma)$ that are $\prec_{\text{lex}}$-minimal.

(Recall that $\Sigma(\Gamma)$ is the set of all $\Theta \subseteq \Omega$ such that $Dab(\Theta)$ is a minimal $Dab$-formula at stage $s$.)
Marking

Marking definition
A line \( l \) with an argument \( A \) and a condition \( \Delta \) is marked iff
- there is a \( \Theta \in \Phi_{\text{lex}}(\Gamma) \) for which \( \Delta \cap \Theta = \emptyset \)
- for each \( \Theta \in \Phi_{\text{lex}}(\Gamma) \) there is a line \( l' \) with argument \( A \) and a condition \( \Delta' \) for which \( \Delta' \cap \Theta = \emptyset \).

Final Derivability
as before

Back to our example
prove \( p \land q \) from \( \Gamma = \{ \circ_1 p, \circ_2 q, \circ_3 r, \neg p \lor \neg q \lor \neg r \} \)

Is there a problem with strong reassurance?

Let \( \Delta \) be a choice set of \( \Sigma(\Gamma) \). We show that there is a \( \prec_{\text{lex}} \)-minimal choice set \( \Delta' \) of \( \Sigma(\Gamma) \) for which \( \Delta' \subseteq \Delta \).

Let \( \Delta \) be a choice set of \( \Sigma(\Gamma) \). We show that there is a \( \prec_{\text{lex}} \)-minimal choice set \( \Delta' \) of \( \Sigma(\Gamma) \) for which \( \Delta' \subseteq \Delta \).

The construction of \( \Delta' \)
- Let \( \Sigma_1(\Gamma) \) be the set of all \( \Theta \subseteq \Omega_1 \) such that \( \Theta \in \Sigma(\Gamma) \).
- We know that \( \Delta_1 \) is a choice set of \( \Sigma_1(\Gamma) \).
- We also know that there is a \( \Delta'_1 \subseteq \Delta_1 \) that is a minimal choice set of \( \Sigma_1(\Gamma) \) (see one of the previous sessions).
- Let \( \Sigma_2(\Gamma) \) the set of all \( \Theta \setminus \Omega_1 \) for which \( \emptyset \neq \Theta \setminus \Omega_1 \subseteq \Omega_2 \), \( \Delta'_1 \cap \Theta = \emptyset \), and \( \Theta \in \Sigma(\Gamma) \).
  - If \( \Delta'_1 \subseteq \Delta_1 \) let \( \Delta'_2 \) be any minimal choice set of \( \Sigma_2(\Gamma) \).
  - Otherwise, \( \Delta_2 \) is a choice set of \( \Sigma_2(\Gamma) \). Let \( \Delta'_2 \) be a minimal choice set of \( \Sigma_2(\Gamma) \) such that \( \Delta'_2 \subseteq \Delta_2 \).
- Let \( \Sigma_3(\Gamma) \) the set of all \( \Theta \setminus (\Omega_1 \cup \Omega_2) \) for which \( \emptyset \neq \Theta \setminus (\Omega_1 \cup \Omega_2) \subseteq \Omega_3 \), \( (\Delta'_1 \cup \Delta'_2) \cap \Theta = \emptyset \), and \( \Theta \in \Sigma(\Gamma) \).
  - If \( \Delta'_1 \cup \Delta'_2 \subseteq \Delta_1 \cup \Delta_2 \) let \( \Delta'_3 \) be any minimal choice set of \( \Sigma_3(\Gamma) \).
  - Otherwise, \( \Delta_3 \) is a choice set of \( \Sigma_3(\Gamma) \). Let \( \Delta'_3 \) be a minimal choice set of \( \Sigma_3(\Gamma) \) such that \( \Delta'_3 \subseteq \Delta_3 \).
- We proceed similarly for all \( i \geq 4 \).
- Let \( \Delta' = \bigcup_{i \geq 1} \Delta'_i \).
How does this help with reassurance? Abnormal parts and choice sets

From abnormal parts to choice sets

Where \( M \in \mathcal{M}(\Gamma) \), \( \text{Ab}(M) \) is a choice set of \( \Sigma(\Gamma) \).

From choice sets to abnormal parts

Where \( \Delta \) is a choice set of \( \Sigma(\Gamma) \) there is a model \( M \in \mathcal{M}(\Gamma) \) such that \( \text{Ab}(M) \subseteq \Delta \).

Proof

- Suppose there is no model \( M \in \mathcal{M}(\Gamma) \) for which \( \text{Ab}(M) \subseteq \Delta \).
- Hence, \( \Gamma \cup (\Omega \setminus \Delta) \) is LLL-trivial.
- Hence, \( \Gamma \cup (\Omega \setminus \Delta) \models A \) for any \( A \in \Omega \setminus \Delta \).
- By the compactness of LLL there is a finite subset of \( \Theta \subseteq (\Omega \setminus \Delta) \) such that \( \Gamma \cup \Theta \models A \).
- By the deduction theorem, \( \Gamma \models \Theta \vdash A \) and hence \( \Gamma \models \text{Dab}(\Theta \cup \{A\}) \).
- Hence, there is a minimal \( \Theta' \subseteq \Theta \cup \{A\} \) such that \( \Gamma \models \text{Dab}(\Theta') \).
- (Note that \( \Theta' \subseteq \Sigma(\Gamma) \).)
- Since \( \Delta \) is a choice set of \( \Sigma(\Gamma) \) and \( \Delta \cap \Theta = \emptyset \) this is a contradiction.

Christian Straßer
Strong Reassurance

Where \( M \in \mathcal{M}(\Gamma) \) there is a \( M' \in \mathcal{M}_p(\Gamma) \) for which \( \text{Ab}(M') \subseteq \text{Ab}(M) \).

**Proof**

- Let \( M \in \mathcal{M}(\Gamma) \).
- Hence, \( \text{Ab}(M) \) is a choice set of \( \Sigma(\Gamma) \).
- Thus, there is a \( \Delta \) that is a \( \prec_{\text{lex}} \)-minimal choice set of \( \Sigma(\Gamma) \) for which \( \Delta \prec_{\text{lex}} \text{Ab}(M) \).
- Hence, there is a \( M' \in \mathcal{M}(\Gamma) \) for which \( \text{Ab}(M') \subseteq \Delta \).
- Assume \( \text{Ab}(M') \subset \Delta \). Then \( \text{Ab}(M') \prec \Delta \) (try to see this!). Since \( \text{Ab}(M') \) is a choice set of \( \Sigma(\Gamma) \) this is a contradiction. Hence, \( \text{Ab}(M') = \Delta \).
- Hence, \( M' \in \mathcal{M}_p(\Gamma) \).